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Mathematical Reviews

Vol. 6, No. 8 SEPTEMBER, 1945 Pages 197-224

FOUNDATIONS

Fitch, Frederic B. Representations of calculi. J. Symbolic Logic 9, 57–62 (1944).

In a previous paper [J. Symbolic Logic 7, 105–114 (1942); these Rev. 4, 125] Fitch developed a system of logic K, similar to systems due to Church, Curry, and Rosser. He conjectured that the logic K is basic in the sense that every system of logic, including the ramified version of the system of Principia Mathematica, is definable in it. The syntactical counterpart of this conjectured result is proved in the present paper, utilizing the method of Gödel representation. Fitch proves theorems to the effect that the reflection of every general recursive function is represented in K; that every recursively enumerable class of numerals is represented in K, and hence also the class of numerals denoting the Gödel numbers of any calculus; that the relation between some expression of a calculus and the numeral denoting its Gödel number is represented in K, and hence that every calculus (that is, every class of expressions having a recursively enumerable class of integers as its Gödel representation) is represented in K. Thus the logic K is basic in the intended sense. *R. M. Martin* (Chicago, Ill.).

Hempel, Carl G. Studies in the logic of confirmation. I. Mind 54, 1–26 (1945). [MF 11769]
Hempel, Carl G. Studies in the logic of confirmation. II. Mind 54, 97–121 (1945). [MF 12349]

Finsler, Paul. Gibt es unentscheidbare Sätze? Comment. Math. Helv. 16, 310–320 (1944).

The author maintains that there do not exist "absolutely undecidable" sentences. His argument rests on a distinction between the "explicit" and "implicit" meanings of sentences.

It is hard to understand what content the author's thesis possesses. It is, of course, obvious that every sentence is decidable in some formal system, since we can construct a system having the given sentence as an axiom. The author seems to have in mind some idealist (as opposed to formal) conception of provability, but the nature of this conception is never made clear. *J. C. C. McKinsey.*

Wernick, William. Distributive properties of set operators. Bull. Amer. Math. Soc. 51, 120–125 (1945). [MF 11826]

This paper is concerned with elementary problems regarding the mutual relations of certain properties of set operators. Fourteen such properties are considered including, for example, the property of being distributive with respect to set-theoretical addition. The possibilities are explored as to what combinations of these properties can be realized, what properties imply others, and the like.
J. C. C. McKinsey (Bozeman, Mont.).

Chwistek, L. Sur les fondements de la sémantique. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 187–194 (1943). (Russian. Georgian and French summaries) [MF 11691]

En nous basant sur le système de la sémantique élémentaire, nous construisons une hiérarchie de systèmes laquelle embrasse du même temps la métamathématique, la théorie élémentaire des ensembles et une partie de mathématiques classiques. Ensuite nous passons au système plus vaste, dit système de Hetper, que nous complétons en introduisant les types transfinis de propositions. Dans ce système nous obtenons une théorie des ensembles équivalente à la théorie classique dans laquelle interviennent l'axiome de Zermelo et l'hypothèse du continuum comme des simples théorèmes. *Author's summary.*

Chwistek, L. Sur les notions fondamentales de la théorie de nombres généralisés. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 507–514 (1943). (Russian. Georgian and French summaries) [MF 11692]

Nous envisageons des suites de nombrés réels, des suites de ces suites, etc. De cette façon nous obtenons de suites à n étages. Par une généralisation facile nous passons aux suites à ω, $\omega+1$, $\cdots$, ξ étages, où ξ est un nombre ordinal transfini plus petit que Ω. Nous traitons nos suites comme des nombres, en effectuant les opérations arithmétiques par membres et en ne tenant compte que des membres possédant le numéro plus grand qu'un entier donné. Ces nombres possèdent des propriétés fondamentales de nombres réels, en nous fournissant en même temps des infiniment petits et des infiniment grands.

Dans le domaine de nombres généralisés nous construisons l'analyse, dans laquelle les fonctions de Baire peuvent être représentées par des fonctions régulières généralisées, qui dans des points réels ne diffèrent d'elles que par une quantité infiniment petite. De cette façon le problème de l'existence de limites a été éliminé de l'analyse classique.

En nous servant de l'analyse généralisée, nous pouvons construire une théorie des opérateurs linéaires, qui nous permet de traiter les matrices finies et infinies comme des cas spéciaux des opérateurs de Dirac. En dehors de ça l'analyse généralisée nous fournit des moyens de recherche, qui se montrent féconds dans la géométrie abstraite et en particulier dans "la géométrie du monde," dans le calcul des probabilités, etc. *Author's summary.*

Chwistek, L. Sur les notions fondamentales de l'analyse généralisée. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 745–752 (1943). (Georgian and Russian. French summary) [MF 11693]

En nous basant sur le corps de nombres généralisés, nous pouvons construire l'analyse généralisée, qui, pardessus des

applications importantes, nous présente des phénomènes mathématiques intéressants que l'on pourrait à peine retrouver dans le domaine de mathématiques courantes. En particulier, elle nous donne la possibilité de construire des fonctions régulières, qui pourtant dans des points réels acquièrent les mêmes valeurs qu'une fonction de Baire d'une classe aussi haute que nous voulons. D'autre part elle nous fournit des moyens de construire une théorie des opérateurs linéaires embrassant dans les mêmes formules les matrices et les opérateurs continus. Enfin elle nous présente des problèmes très abstraits concernant les propriétés des fonctions généralisées discontinues. *Author's summary.*

Sales, Francisco. On the potential infinite and the actual infinite. Revista Mat. Hisp.-Amer. (4) **4**, 159–161 (1944). (Spanish) [MF 12171]

Sancho de San Roman, J. On the existence of nondenumerable sets. Revista Mat. Hisp.-Amer. (4) **4**, 24–30 (1944). (Spanish) [MF 12160]

Moorman, R. H. The influence of mathematics on the philosophy of Leibniz. Nat. Math. Mag. **19**, 131–140 (1944). [MF 11950]

Hempel, C. G. Geometry and empirical science. Amer. Math. Monthly **52**, 7–17 (1945). [MF 11906]

an essay on the

★**Hadamard, Jacques. The Psychology of Invention in the Mathematical Field.** Princeton University Press, Princeton, N. J., 1945. xiii+143 pp. $2.00.

The table of contents reads as follows. I. General views and inquiries. II. Discussions on unconsciousness. III. The unconscious and discovery. IV. The preparation stage. Logic and chance. V. The later conscious work. VI. Discovery as a synthesis. The help of signs. VII. Different kinds of mathematical minds. VIII. Paradoxical cases of intuition. IX. The general direction of research. Final remarks.

★**Polya, G. How to Solve It. A New Aspect of Mathematical Method.** Princeton University Press, Princeton, N. J., 1945. xv+204 pp. $2.50.

ALGEBRA

González del Valle, A. A relation among binomial coefficients, obtained as consequence of an electrical problem. Revista Mat. Hisp.-Amer. (4) **4**, 48–50 (1944). (Spanish) [MF 12162]

By identifying two solutions of an electrical problem, the author obtains an identity equivalent to

$$\sum_s {}_{p+s}C_s \cdot {}_{2p+q}C_{2p+2s+1} = 2^{q-1}\,{}_{p+q-1}C_p.$$

In Netto's "Lehrbuch der Combinatorik" [second ed., Teubner, Leipzig, 1927, p. 253], the formula is proved more directly by equating coefficients of x^{q-1} in

$$(1-x)^{-2p}[1-x^2(1-x)^{-2}]^{-p} = (1-2x)^{-p}.$$

I. Kaplansky (New York, N. Y.).

Hsu, L. Ching-Siur. A combinatorial formula with some applications. Bull. Amer. Math. Soc. **51**, 106–113 (1945). [MF 11824]

The author proves a rather complicated combinatorial formula, which cannot be reproduced because of lack of space. He applies his formula to partitions and to Dirichlet's integral. *P. Erdös* (Ann Arbor, Mich.).

Sah, A. Pen-Tung. A uniform method of solving cubics. Amer. Math. Monthly **52**, 202–206 (1945). [MF 12257]

Bilharz, Herbert. Geometrische Darstellung eines Satzes von Hurwitz für Frequenzgleichungen fünften und sechsten Grades. Z. Angew. Math. Mech. **21**, 96–102 (1941). [MF 12149]

For real equations $f_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ with $a_0 > 0$ and $n = 4$, 5 and 6, the author restates in geometric form the well-known criterion of Hurwitz for all the zeros of $f_n(x) = 0$ to have negative real parts. When $n = 4$, the criterion is shown to be identical with the requirement that the vectors **A** and **B**, where $\mathbf{A} = a_1\mathbf{i} + a_3\mathbf{j}$ and

$$a_1\mathbf{B} = (a_1a_2 - a_0a_3)\mathbf{i} + a_1a_4\mathbf{j},$$

make with the positive x-axis the respective angles α and β with $0 < \alpha < \beta < \pi/2$. If we define the vectors **A** and **B** as $\mathbf{A} = a_0\mathbf{i} + a_2\mathbf{j} + a_4\mathbf{k}$, $\mathbf{B} = a_1\mathbf{i} + a_3\mathbf{j} + a_5\mathbf{k}$ when $n = 5$, and by $\mathbf{A} = a_1\mathbf{i} + a_3\mathbf{j} + a_5\mathbf{k}$, $a_1\mathbf{B} = (a_1a_2 - a_0a_3)\mathbf{i} + (a_1a_4 - a_0a_5)\mathbf{j} + a_0a_1\mathbf{k}$ when $n = 6$, then the criterion is shown equivalent with the requirements that the two vectors **A** and **B** lie in the octant $x > 0$, $y > 0$, $z > 0$ and that the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ lies in that portion of the octant $x < 0$, $y > 0$, $z < 0$ which is interior to the cone $xz - y^2 = 0$. *M. Marden* (New York, N. Y.).

Todd, J. A. The 'odd' number six. Proc. Cambridge Philos. Soc. **41**, 66–68 (1945). [MF 12377]

In general (that is, when $n \neq 6$), if a rational function ϕ_1 of the roots $\alpha_1, \cdots, \alpha_n$ of an algebraic equation $f(x) \equiv x^n + a_1x^{n-1} + \cdots + a_n = 0$ has but n conjugates when the $n!$ permutations of the symmetric group Σ_n of degree n are applied to the n roots α_i, the n conjugates of ϕ_1 can be so ordered that each ϕ_i can be expressed as the same rational function ϕ of a different one of the roots ($\phi_r = \phi(\alpha_r)$). The coefficients in ϕ_i and ϕ are assumed to be in an algebraic field $K(a_1, \cdots, a_n)$ obtained by adjoining the n indeterminates a_i to the field K of complex numbers. For the single case $n = 6$, it is not true that the only subgroups which are of index n in Σ_n are the n conjugate subgroups each leaving a different symbol fixed. On the contrary, for $n = 6$, there is a second set of 6 conjugate subgroups of Σ_6. One of these subgroups leaves fixed the function

$$\phi_I = (14, 25, 36)(16, 24, 35)(13, 26, 45)(12, 34, 56)(15, 23, 46),$$

where $(14, 25, 36)$ stands for $\alpha_1\alpha_4 + \alpha_2\alpha_5 + \alpha_3\alpha_6$, etc. This function ϕ_I does not belong to any of the fields obtained by adjoining only a single root α_i to $K(a_1, \cdots, a_n)$. Its six conjugates may be obtained by applying to ϕ_I the six permutations of one of the symmetric subgroups on three symbols. *J. S. Frame* (East Lansing, Mich.).

Ghosh, N. N. A note on Hermitian matrix. Bull. Calcutta Math. Soc. **36**, 87–90 (1944). [MF 11839]

A simple Hermitian matrix generated by a single-column matrix H is, by definition, the product of H and the single-row matrix which is the transpose of the complex conjugate of H. Given a Hermitian matrix A, the author constructs (using the minors of different orders of A) sequences of single-column matrices and shows that A can be presented as a linear combination of the simple matrices generated by

the matrices of such a sequence. He also considers the vector space associated with A and the canonical resolution of A in terms of eigenvectors and eigenvalues of A.

G. Y. Rainich (Ann Arbor, Mich.).

Wagner, Edwin. Die Elementarteiler eines Polynoms in einer Matrix. Math. Z. 49, 328–338 (1944). [MF 12001]

The author determines the elementary factors of a matrix which is a polynomial in a matrix A over a field of characteristic zero. These have been determined before at least twice [N. H. McCoy, Amer. J. Math. 57, 491–502 (1935) and J. Williamson, Amer. J. Math. 58, 747–758 (1936)]. The author applies his results to the consideration of the matrix equation $f(X)=A$.

J. Williamson (Flushing, N. Y.).

Reisch, Paul. Neue Lösungen der Funktionalgleichung für Matrizen $\Phi(X)\Phi(Y)=\Phi(XY)$. Math. Z. 49, 411–426 (1944). [MF 11995]

This paper is based on a paper by O. Perron [Math. Z. 48, 136–172 (1942); these Rev. 5, 30] in which he considered the functional equation

$$\Phi(X)\Phi(Y)=\Phi(XY), \tag{1}$$

where Φ is a ν-rowed square matrix and X and Y are n-rowed square matrices. Let $X=(x_{pq})$ and let A be the matrix $\partial\Phi(X)/\partial x_{11}$ calculated when $X=E$, the identity matrix of order n. The author proves that if the characteristic numbers of A are all distinct there exist nontrivial irreducible solutions of (1), analytic in the neighborhood of the origin, only if $n=2$. He determines those solutions which are of the form $|X|^a\Phi_\nu(X)$. The matrix $\Phi_\nu(X)$ is the transpose of the matrix of the transformation on the coefficients of a binary ν-ic induced by a transformation on the variables of the matrix X. He also shows that when $\nu=4$ and $A-\lambda E$ has two quadratic elementary divisors there exist nontrivial irreducible solutions of (1), analytic in the neighborhood of the origin, only if $n=2$. Any such solution is equivalent to

$$\Phi(X)=|X|^a\begin{pmatrix} X & 0 \\ \log|X|X & X\end{pmatrix}.$$

J. Williamson (Flushing, N. Y.).

Stöhr, Alfred. Oszillationstheoreme für die Eigenvektoren spezieller Matrizen. J. Reine Angew. Math. 185, 129–143 (1943). [MF 12106]

Let

$$M(\lambda)=\begin{Vmatrix} g_1(\lambda) & k_1 & 0 & \cdots & 0 \\ k_1 & g_2(\lambda) & k_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & g_N(\lambda)\end{Vmatrix}$$

be a real symmetric matrix of order N, where the k_i are positive constants and each $g_i(\lambda)$ is a monotonic continuous function of λ in $-\infty<\lambda<\infty$. If u is a column vector the author calls a value of λ satisfying $M(\lambda)u=0$ a characteristic value and the corresponding vector u a characteristic vector. He shows that there are exactly N characteristic values and that no two of them are equal. He proves that, if the characteristic values $\lambda^{(i)}$ are arranged in increasing order and if $u^{(i)}$ is the characteristic vector corresponding to $\lambda^{(i)}$, then there are exactly $m-1$ changes in sign in the sequence formed by the signs of the elements in the mth row or the mth column of the matrix $\{u_j^{(i)}\}$.

J. Williamson (Flushing, N. Y.).

Conway, A. W. Quaternions and matrices. Proc. Roy. Irish Acad. Sect. A. 50, 98–103 (1945). [MF 12196]

After a short historical introduction the author proceeds to determine quaternion matrices which obey the same commutation rules as the Duffin-Kemmer matrices [Duffin, Phys. Rev. (2) 54, 1114 (1938); Kemmer, Proc. Roy. Soc. London. Ser. A. 173, 91–116 (1939)]. He generalizes the linear quaternion functions in terms of which the Eddington matrices may be expressed [Conway, Proc. Roy. Soc. London. Ser. A. 162, 145–154 (1937)] to the functions $f_r=\frac{1}{2}(\alpha_r(\ )+(\ )\alpha_r')$ and $g_r=\frac{1}{2}(\alpha_r(\ )-(\ )\alpha_r')$, where $w_r+\alpha_r$ and $w_r+\alpha_r'$ are two sets of four quaternions with the same scalar parts w_r and each set is an orthogonal set of four vectors, $r=1, 2, 3, 4$. He obtains his desired result by showing that the matrices

$$\Gamma_r=\begin{pmatrix} 0 & 0 & f_r & 0 \\ 0 & 0 & w_r & -g_r \\ -f_r & w_r & 0 & 0 \\ 0 & g_r & 0 & 0\end{pmatrix}$$

satisfy the equations $\Gamma_r\Gamma_s\Gamma_t+\Gamma_t\Gamma_s\Gamma_r=\delta_{rs}\Gamma_t+\delta_{ts}\Gamma_r$ $(r, s, t=1, 2, 3, 4)$.

J. Williamson (Flushing, N. Y.).

Murnaghan, Francis D. A modern presentation of quaternions. Proc. Roy. Irish Acad. Sect. A. 50, 104–112 (1945). [MF 12197]

This paper is practically identical with the author's paper in Scripta Math. 10, 37–49 (1944); these Rev. 6, 142.

J. Williamson (Flushing, N. Y.).

Colthurst, J. Riversdale. The icosian calculus. Proc. Roy. Irish Acad. Sect. A. 50, 112–121 (1945). [MF 12198]

In this paper the author gives an account of Hamilton's investigations on the icosian calculus, a calculus involving the three symbols ι, κ, λ, where $\iota^2=\kappa^3=\lambda^5=1$, $\lambda=\iota\kappa$ and $\iota\kappa\neq\kappa\iota$, and so called since these symbols may be interpreted as having reference to the passage from face to face of an icosahedron. As the author had access to Hamilton's letters and notebooks his account contains much that has not previously appeared in print and justifies his concluding paragraph: "Hamilton's work on the icosian calculus not only affords a notable exemplification of the far-reaching scope of his conceptions but also illustrates what P. G. Tait describes as one of the peculiar characteristics of his mind —'never to be satisfied with a general understanding of a question: he pursued it until he knew it in all its details.'"

J. Williamson (Flushing, N. Y.).

Abstract Algebra

MacDuffee, C. C. On the composition of algebraic forms of higher degree. Bull. Amer. Math. Soc. 51, 198–211 (1945). [MF 12089]

[Lecture before the American Mathematical Society.] If f, g and h are homogeneous forms of degree k with coefficients in a commutative ring $\mathfrak{R}$ with unit element, such that $f(x)=g(y)\cdot h(z)$ is an identity in virtue of

$$x_k=\sum_{i,j} c_{ijk}y_iz_j, \quad i, j, k=1, 2, \cdots, n,$$

then $f(x)$ is composite and g and h are composable. The theory of composition is discussed only for principal ideal rings $\mathfrak{R}$. This lecture reviews previous results of the author [Amer. J. Math. 64, 646–652 (1942); these Rev. 4, 70],

notably the following: if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of an integral domain of a finite algebra over the quotient field of $\mathfrak{R}$, then $n^{\mathfrak{b}}(\mathfrak{a})\cdot n(\mathfrak{b})=n(\mathfrak{c})$, where $n^{\mathfrak{b}}(\mathfrak{a})$ is the absolute value of the determinant of the matrix which corresponds to the class of $\mathfrak{b}$ and $\mathfrak{a}\times\mathfrak{b}=\mathfrak{c}$. This result applies directly to give a composition of the type described above. When $\mathfrak{R}$ is the ring of rational integers, these results connect with the Dedekind-Weber theorem. Also, the methods are applied to a noncommutative ring $\mathfrak{R}$ to produce a somewhat unorthodox composition. *B. W. Jones* (Ithaca, N. Y.).

Jacobson, N. Structure theory of simple rings without finiteness assumptions. Trans. Amer. Math. Soc. 57, 228–245 (1945). [MF 12132]

This is a theory of rings which are simple, which are not zero-rings and which possess either minimal right ideals or maximal right ideals. In either case, the ring admits an irreducible faithful representation by endomorphisms of an additive group $\mathfrak{G}$. By Schur's lemma, the endomorphisms of $\mathfrak{G}$ which commute with those which represent the elements of the ring form a division algebra $\mathfrak{D}$, and $\mathfrak{G}$ may be considered as a vector space over $\mathfrak{D}$. The next step is to prove a generalization of Burnside's theorem to the effect that an irreducible ring $\mathfrak{o}$ of endomorphisms of $\mathfrak{G}$ over $\mathfrak{D}$ is dense in the ring of all endomorphisms, in the sense that, $\{u_1, \cdots, u_k\}$ and $\{v_1, \cdots, v_k\}$ being any two systems of k elements of $\mathfrak{G}$ and the u's being linearly independent over $\mathfrak{D}$, there exists an $x\epsilon\mathfrak{o}$ such that $x(u_i)=v_i$ $(1\leqq i\leqq k)$. The ring $\mathfrak{o}$ will contain minimal right ideals if and only if its transformations are finite-valued (that is, each of them maps $\mathfrak{G}$ upon a space of finite dimension over $\mathfrak{D}$); in this case, it is proved that the ring admits only one irreducible nontrivial representation (except for isomorphism). In the case where $\mathfrak{D}$ is a field, it is shown that $\mathfrak{o}$, if composed of finite-valued transformations, is central simple and that it is locally finite over $\mathfrak{D}$ (that is, every finite subset of $\mathfrak{o}$ generates a finite algebra). Still assuming that $\mathfrak{D}$ is a field, any isomorphism between two dense rings of finite-valued endomorphisms of $\mathfrak{G}$ over $\mathfrak{D}$ can be extended to an automorphism of the full ring $\mathfrak{L}$ of endomorphisms of $\mathfrak{G}$. Moreover, every automorphism of $\mathfrak{L}$ is inner.

Some applications are made to the theory of nonassociative rings. Let $\mathfrak{A}$ be a nonassociative ring which is simple and not a zero ring. The multiplication ring $\mathfrak{M}$ of $\mathfrak{A}$ is defined to be the ring of endomorphisms of the additive group of $\mathfrak{A}$ which is generated by the left- and right-multiplication by elements of $\mathfrak{A}$. Those endomorphisms of $\mathfrak{A}$ which commute with every operation of $\mathfrak{M}$ form a ring, the multiplication centralizer of $\mathfrak{A}$. This ring is proved to be a field; $\mathfrak{A}$ may be considered as a vector space over this field, and $\mathfrak{M}$ then becomes a dense ring of endomorphisms of this vector space. The center of $\mathfrak{A}$ is defined to be the set of elements c of $\mathfrak{A}$ such that $cx=xc$, $(xy)c=x(yc)=(xc)y$ for every x and y in $\mathfrak{A}$. If this center is not $\{0\}$, $\mathfrak{A}$ has a unit element and the multiplication centralizer consists exactly of the multiplications by elements of the center. *C. Chevalley* (Princeton, N. J.).

Teichmüller, Oswald. Über die partielle Differentiation algebraischer Funktionen nach einem Parameter und die Invarianz einer gewissen Hauptteilsystemklasse. J. Reine Angew. Math. 186, 49–57 (1944). [MF 12101]

The author develops the theory of abstract differentiation in rings with an application to algebraic function fields of one variable. He obtains another proof of a theorem on the principal parts of differentials in a function field. [For the background of the new algebraic approach, see O. Teichmüller, same J. **185**, 1–11 (1943); M. Eichler, same J. **185**, 12–13 (1943); these Rev. **5**, 37, 38.] *O. F. G. Schilling* (Chicago, Ill.).

Delaunay, B., and Faddeieff, D. Investigations in the geometry of the Galois theory. Rec. Math. [Mat. Sbornik] N.S. 15(57), 243–284 (1944). (Russian. English summary) [MF 12287]

The authors first derive the fundamental theorems of Galois theory by the geometrical method explained in an article by Delaunay [Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 52–62; these Rev. **3**, 101]. (The lattices described there are replaced by subspaces in which the coefficients range over an arbitrary field rather than over the rational integers.) The results are mostly well known, but their derivation by purely geometric methods is of interest.

The main part of the paper concerns the following imbedding problem. Let the finite group $\mathfrak{G}$ have an invariant subgroup $\mathfrak{N}$, with $\mathfrak{G}/\mathfrak{N}\cong\mathfrak{F}$. Then $\mathfrak{G}$ is a group extension of $\mathfrak{N}$ by means of a set of automorphisms of $\mathfrak{N}$ and a Schreier factor system $C_{\sigma,\tau}$ of elements of $\mathfrak{N}$ [Zassenhaus, Lehrbuch der Gruppentheorie, vol. 1, Teubner, Leipzig-Berlin, 1937, chap. III, §6]. Let fields k and R be given so that k is a normal extension of R with Galois group $\mathfrak{F}$; the problem is to construct a field $K\supset k$ such that K is normal over R with Galois group $\mathfrak{G}$, and its subfield k belongs to the subgroup $\mathfrak{N}$. The authors prove that such a field K always exists in the case of Abelian $\mathfrak{N}$ and unit factor system ($C_{\sigma,\tau}=1$ for all σ, τ); R can be any finite algebraic extension of the rational field. Conditions for the solubility of the problem in more general cases are also given. Most of the recent work on this problem [see H. Reichardt, J. Reine Angew. Math. **177**, 1–5 (1937) and the references given there] uses class field theory; thus it is noteworthy that the construction by means of Delaunay's geometric Galois theory is entirely elementary. *G. Whaples* (Philadelphia, Pa.).

Wagner, Edwin. Über Shodasche Matrizen und Polynome in einer Matrix. Math. Z. 49, 517–537 (1944). [MF 11991]

Let $\mathfrak{K}$ be a field of characteristic 0, t an indeterminate, and let $a_0(t), a_1(t), \cdots, a_{s-1}(t)$ be a system of elements of $\mathfrak{K}[t]$, each with leading coefficient 1, and $a_i(t)$ a divisor of $a_{i-1}(t)$. A square matrix of order s, whose elements $g_{ij}(t)$ are elements of $\mathfrak{K}[t]$, where $g_{ij}(t)$ is divisible by $a_j(t)/a_i(t)$ for $j<i$, is called a Shoda matrix [K. Shoda, Math. Z. **29**, 696–712 (1929)]. An S-ideal is an ideal (other than the unit ideal) in the polynomial ring $\mathfrak{K}[x, t]$, which contains a polynomial $u(t)$ depending only on t and also a polynomial $v(t, x)$ in which the coefficient of the highest power of x does not depend on t. The author shows that each S-ideal in the ring $\mathfrak{K}[x, t]$ is uniquely determined by a Shoda matrix, and conversely. He then considers elementary divisor groups of the second kind [extension of a notion of Krull, S.-B. Heidelberger Akad. Wiss. **1926**, no. 1 (1926)] defined as $\mathfrak{K}$-modules of rank n having as operator domain $\mathfrak{K}[A, B]$, where A and B are commutative operators not in $\mathfrak{K}$. Such a $\mathfrak{K}$-module is associated with a class of Shoda matrices. Various theorems are obtained but special attention is given to the case in which $B=f(A)$ or $A=g(B)$. The final results have to do with the elementary divisors of a polynomial in a given matrix. *N. H. McCoy.*

Schilling, O. F. G. Noncommutative valuations. Bull. Amer. Math. Soc. **51**, 297–304 (1945). [MF 12272]

The author extends the fundamental results of valuation theory to noncommutative value groups. With appropriate modifications, the ideal theory, arithmetic and decomposition of valuation carry through. In order to have a noncommutative value group, a skew field must be transscendental over its center; examples of such fields are furnished by certain generalized crossed products.

I. Kaplansky (New York, N. Y.).

Chevalley, C. On groups of automorphism of Lie groups. Proc. Nat. Acad. Sci. U.S.A. **30**, 274–275 (1944). [MF 11041]

If $\mathfrak{g}$ is a Lie algebra, a derivation D in $\mathfrak{g}$ is a linear mapping in $\mathfrak{g}$ such that $D([X, Y]) = [DX, Y] + [X, DY]$. It is well known that the set of these mappings is a Lie algebra $\mathfrak{D}(\mathfrak{g})$; $\mathfrak{D}(\mathfrak{g})$ is the Lie algebra of the group of automorphisms of a Lie group if $\mathfrak{g}$ is the Lie algebra of the group. For any $\mathfrak{g}$, $\mathfrak{D}(\mathfrak{g})$ contains an ideal $\mathfrak{J}$ consisting of the inner derivations $X \to [X, A]$, and $\mathfrak{J}$ is isomorphic to the difference algebra $\mathfrak{g} - \mathfrak{c}$, $\mathfrak{c}$ the center of $\mathfrak{g}$. A Lie algebra is called complete if $\mathfrak{J} = \mathfrak{D}$. Let $\mathfrak{D}_n(\mathfrak{g})$ be defined inductively as $\mathfrak{D}(\mathfrak{D}_{n-1}(\mathfrak{g}))$, with $\mathfrak{D}_0(\mathfrak{g}) = \mathfrak{g}$. The author announces the following result. Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0 and suppose that $\mathfrak{c} = 0$. Then there exists an index n such that $\mathfrak{D}_n(\mathfrak{g})$ is complete. The proof is sketched.

N. Jacobson (Baltimore, Md.).

THEORY OF GROUPS

Lorenzen, Paul. Ein Beitrag zur Gruppenaxiomatik. Math. Z. **49**, 313–327 (1944). [MF 12002]

A great variety of properties have been used as axioms for group theory. In this paper they are collected and systematized; the author's list contains more than 40 items. The main object of this investigation is the establishment of all the shortest complete sets of axioms of group theory contained in this system of group theoretical properties. These are deduced; the author obtains more than 20 systems, containing 3 or 4 axioms each, any one of which may serve as a definition of the group concept.

R. Baer (Urbana, Ill.).

Sadovsky, L. Structural isomorphisms of free groups and of free products. Rec. Math. [Mat. Sbornik] N.S. **14(56)**, 155–173 (1944). (Russian. English summary) [MF 12295]

The totality of subgroups of a group forms a lattice; two groups are structurally isomorphic if there exists an order-preserving one-to-one correspondence between their subgroups. Some groups have been investigated from this point of view, but even the groups structurally isomorphic to the Abelian groups are not completely known. The following theorems are proved. Every free group is determined by its structure and if it is not infinitely cyclic then each of its structural isomorphisms is induced by a single group isomorphism. The same is true of a locally free group if its rank is greater than one. The structural isomorphism f of a free product $A*F$, where F is free noncyclic, carries A into a group A^f isomorphic with it and the structural isomorphism of A is a consequence of at least one group isomorphism. The same is true if one factor is arbitrary and the other is of rank at least 3. If G is a free product and does not possess a finite system of generating elements, then every structural isomorphism is induced by just one group isomorphism. If one or both factors are of rank less than 3 then it is still true that any structurally isomorphic group is generated by groups isomorphic to the factors, but it is still unknown whether it is the free product of these groups.

M. S. Knebelman (Pullman, Wash.).

Dietzmann, A. P. On the criteria of non-simplicity of groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 89–91 (1944). [MF 12028]

Let Γ be a group and H and Q subgroups such that the double coset decomposition $\Gamma = Hg_1Q + \cdots + Hg_kQ$ is finite; let K be a normal subgroup of H with H/K Abelian and such that $H \cap g_iQg_i^{-1} \subset K$ for $i = 1, \cdots, k$; let $\bar{N}$ be the normaliser of Q in Γ and N a subgroup of $\bar{N}$. The main results announced are as follows: if $n \varepsilon N$ and $g_in = h_i(n)\mathfrak{g}_iq_i$, for $i = 1, \cdots, k$, where $h_i(n) \varepsilon H$, $q_i \varepsilon Q$ and $\mathfrak{g}_1, \cdots, \mathfrak{g}_k$ is some permutation of the residues $g_1, \cdots, g_k$, then the correspondence $n \to h_1 \cdots h_kK$ defines a homomorphism of N upon a subgroup of H/K which is independent of the particular choice of the residues g_i; if $h_1 \cdots h_kK \neq K$, then N contains a normal subgroup N_1 not containing n; if N and H/K are finite the order of N_1 is a multiple of the greatest divisor of the order of N which is relatively prime to the order of H/K. The author deduces four theorems from which, on taking Γ to be finite and Q the unit element, several well-known criteria for the nonsimplicity of finite groups follow.

S. A. Jennings (Vancouver, B. C.).

Tchernikow, S. On the theory of locally soluble groups. Rec. Math. [Mat. Sbornik] N.S. **13(55)**, 317–333 (1943). (Russian. English summary) [MF 11656]

For definitions see the author's two previous papers on the same topic [same Rec. N.S. **7(49)**, 35–64 (1940); **8(50)**, 377–394 (1940); these Rev. **2**, 5, 217]. From the author's résumé we quote the following theorems. (1) A locally finite group possesses a soluble set if and only if it is locally soluble. (2) A locally finite group possesses a principal periodic set if and only if it is locally soluble, or if it is locally quasi-special. (3) It possesses a central set if and only if it is locally special. (4) The commutant of any group possessing a principal periodic set possesses a central set.

M. S. Knebelman (Pullman, Wash.).

Tchounikhin, Irène, et Tchounikhin, S. A. Sur les groupes p-décomposables. Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 325–342 (1944). (Russian. French summary) [MF 12289]

A preliminary announcement of the results of this paper has appeared in C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 43–45 (1943); these Rev. **5**, 143.

Markoff, A. On unconditionally closed sets. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 180–181 (1944). [MF 12033]

The author is concerned with the topological implications of algebraic processes in a group. If G is a group, a set $A \subset G$ is unconditionally closed if it is closed relative to any topology which may be defined in G (under which G is a topological group). A system for constructing unconditionally closed sets is the following. For a finite set of elements $a_i \varepsilon G$ and integers α_i, $i = 1, \cdots, m$, the set of all x

such that $a_i x^{\epsilon_i}=1$ is unconditionally closed. A set is called algebraic if it is the intersection of finite unions of sets of the above type. The smallest algebraic set containing a given set A is its algebraic closure. The principal result is that, if A is an arbitrary subset of a countable group G, then there is a topology of G under which $\bar{A}$ is the algebraic closure of A. The author raises the following question: is there a countable group admitting only the discreet topologization or, equivalently, may the complement of the unit element in a group be algebraically closed?

J. L. Kelley (Aberdeen, Md.).

Weil, André. A correction to my book on topological groups. Bull. Amer. Math. Soc. **51**, 272–273 (1945). [MF 12266]

The author proves a theorem which has the following consequence. If a factor-group of a compact metrizable group G is a Lie group of dimension n, then G itself has dimension at least n. The analogous statement for arbitrary groups and factor groups (which was used implicitly in the author's book [L'intégration dans les groupes topologiques et ses applications, Hermann, Paris, 1940; these Rev. **3**, 198] is still unproved. The above weak form is sufficient for the results of the book on the dimensions of compact groups and locally compact Abelian groups.

H. Samelson (Syracuse, N. Y.).

Levi, F. W. Notes on group theory. III. Homogeneous representation of groups. J. Indian Math. Soc. (N.S.) **8**, 44–56 (1944). [MF 12246]

[Notes I and II appeared in the same J. (N.S.) **8**, 1–9 (1944); these Rev. **6**, 40.] A pencil (Schar) has been defined by H. Prüfer [Math. Z. **20**, 165–187 (1924), in particular, p. 170] and R. Baer [J. Reine Angew. Math. **160**, 199–207 (1929), in particular, p. 202] as a set S of elements which is closed under an operation $ab^{-1}c=d$ for a,b,c,d in S; this makes pencils essentially the same as cosets of a group modulo a subgroup. For this defining operation the author substitutes a relation (a,b,c,d) which may be interpreted as follows: (a,b,c,d) is true if, and only if, $ab^{-1}c=d$. It is shown that the following postulates suffice to define a pencil: (1) given a,b,c in S, there exists one and only one d in S such that (a,b,c,d) is true; (2) (a,a,b,b) and (a,b,b,a) are true; (3) (a,b,c,d) and (b,f,g,c) imply (f,a,d,g); (4) (a,b,c,d) and (d,c,f,g) imply (g,f,b,a). These postulates imply, in particular, the following rules. (1′) To any three of the elements a,b,c,d there exists one and only one fourth one such that (a,b,c,d) is true; (5) (a,b,c,d) implies (b,a,d,c), (d,c,b,a) and therefore (c,d,a,b). The importance of these two properties stems from the fact that (1′) together with (4) and parts of (5) may serve as a definition for a quasi-pencil which is related to quasi-groups in a way similar to that in which pencils are related to groups. *R. Baer.*

Levi, F. W. On semigroups. Bull. Calcutta Math. Soc. **36**, 141–146 (1944). [MF 12413]

A semigroup S is called an R-semigroup if it satisfies the "axiom of refinement," which states that, if A, C and A', C' are any two distinct pairs of elements of S such that $AC=A'C'$, then there exists an element B of S satisfying at least one of the two pairs of conditions $A'=AB$, $C=BC'$ or $A=A'B$, $C'=BC$. It follows from this axiom that any two factorizations $A_1A_2\cdots A_m$, $B_1B_2\cdots B_n$ of the same element α can both be obtained by "bracketing" from a refined factorization $\alpha=C_1C_2\cdots C_r$. A subsemigroup N of S is said to be normal if it satisfies the condition that if any two of the three elements $\alpha\beta\gamma$, $\alpha\gamma$, β belong to N then all three belong to N. Cosets of a normal subsemigroup N of an R-semigroup R are defined and are shown to form an R-semigroup homomorphic to R, having N as unit element. If R' is the semigroup generated by all left-hand factors of elements of R then N is normal in R' and R'/N is a group.

A special type of R-semigroup is the free semigroup. Free semigroups are defined axiomatically and their properties are investigated. The problem of representing a group in the form F/N is dealt with, where F is a free semigroup and N a normal subsemigroup of F. *D. C. Murdoch.*

Krasner, Marc. La caractérisation des hypergroupes de classes et le problème de Schreier dans ces hypergroupes: errata. C. R. Acad. Sci. Paris **218**, 483–484 (1944). [MF 12116]

Corrections to a previous paper by the author [same C. R. **212**, 948–950 (1941); these Rev. **3**, 37] are listed. For details see the following review. *D. C. Murdoch.*

Krasner, Marc. Rectifications à ma note précédente et quelques nouvelles contributions à la théorie des hypergroupes. C. R. Acad. Sci. Paris **218**, 542–544 (1944). [MF 12122]

[Cf. the preceding review.] A hypergroup$_D$ is a hypergroup which is isomorphic to the hypergroup of right cosets of a group $\mathfrak{G}$ with respect to a subgroup $\mathfrak{g}$. If $\mathfrak{H}\cong\mathfrak{G}/\mathfrak{g}$ then $(\mathfrak{G},\mathfrak{g})$ is called a representation of $\mathfrak{H}$. This representation is irreducible if $\mathfrak{g}$ has no proper subgroup which is invariant in $\mathfrak{G}$. Theorem II of the author's previous note, which purported to give necessary and sufficient conditions that a finite hypergroup be a hypergroup$_D$, and its corollary that a finite hypergroup$_D$ has a unique irreducible representation, are shown to be false by means of a counterexample. It is also pointed out that the construction given in the same note for all extensions of one hypergroup$_D$ by another only gives some of these extensions and the adjustment in the construction necessary to obtain all such extensions is described. Finally the author states that he has proved a special case of a hypothesis of J. E. Eaton [Duke Math. J. **6**, 101–107 (1940); these Rev. **1**, 164] that every cogroup is a hypergroup$_D$. The complete proof is not given.

D. C. Murdoch (Vancouver, B. C.).

ANALYSIS

Levi, B. A problem in numerical calculation. On the inversion of functions defined by integrals. Application to an integral of the theory of radiation. Math. Notae **4**, 185–212 (1944). (Spanish) [MF 12205]

The problem is to find numerical values of the function $x=x(y)$ defined by the differential equation $dx/dy=(\log x)^{\frac{1}{2}}$ and the boundary condition $y=0$, $x=1$. This is the same as the problem of finding numerical values of the inverse of the function $y(x)$ defined by

$$y(x)=\int_1^x(\log u)^{-\frac{1}{2}}du=\int_0^{\log x}s^{-\frac{1}{2}}e^s ds=2\int_0^{(\log x)^{\frac{1}{2}}}e^{t^2}dt.$$

R. P. Agnew (Ithaca, N. Y.).

Kjellberg, Bo. Ein Momentenproblem. Ark. Mat. Astr. Fys. **29A**, no. 2, 33 pp. (1943). [MF 12010]

An inequality due essentially to Carlson [same Ark. **25B**, no. 1 (1934)] states that

$$(1)\qquad (\int|f|dx)^4<(2\pi)^2\int|f|^2dx\int x^2|f|^2dx,$$

where the integrations are over $(-\infty, \infty)$. This paper discusses principally the problem of extending (1) to spaces of higher dimension. Let R_n denote the n-dimensional space of points $X=(x_1, \cdots, x_n)$; let A_n be the space of points $Q=(\alpha_1, \cdots, \alpha_n)$; write $X^{2Q}=(x_1^2)^{\alpha_1}\cdots(x_n^2)^{\alpha_n}$, and M_Q^2 for the "moment"

$$M_Q^2=\int|f|^2X^{2Q}dX,$$

integration being over R_n. The author first asks when, given a set S of points Q_j in A_n, the finiteness of $M_{Q_j}^2$ for all Q_j in S implies the finiteness of $\int|f|dX$. Let $P=\sum_S X^{2Q_j}$; then a necessary and sufficient condition for $\sum_S M_{Q_j}^2<\infty$ to imply $\int|f|dX<\infty$ is that $\int P^{-1}dX<\infty$; this condition is satisfied if and only if the point $(\frac{1}{2}, \cdots, \frac{1}{2})$ is an interior point of the convex cover K_S of the set S of points Q_j. It follows, in particular, that S must contain at least $n+1$ points if a generalization of (1) involving the $M_{Q_j}^2$ is to be possible. [If $n=1$, the result thus is that $\sum_j\int x^{2\alpha_j}|f|^2dx<\infty$ implies $\int|f|dx<\infty$ if and only if there is an α_j less than $\frac{1}{2}$ and another greater than $\frac{1}{2}$. The author's results, for $n=1$, overlap the integral analogues of theorems on series given by Gabriel [J. London Math. Soc. **12**, 130–132 (1937)], V. Levin [Rec. Math. [Mat. Sbornik] N.S. **3**(45), 341–345 (1938)] and Caton [Duke Math. J. **6**, 442–461 (1940); these Rev. **2**, 75].] When a generalization of (1) involving $M_{Q_i}^2$ $(Q_i\epsilon S)$ is possible, there remains the problem of determining the powers of M_{Q_i} which should enter on the right side and the best constant. The author solves this problem completely when S contains precisely $n+1$ points, and for one case with 2^n points; the results are too complicated to quote here. [Two special cases when $n=2$ will illustrate the results. Either of the following expressions

$$\Gamma(\tfrac{1}{4})\{\tfrac{1}{2}\pi\textstyle\int\int f^2\}^{\frac{1}{4}}\{\int\int x^4f^2\int\int y^4f^2\}^{\frac{1}{8}},$$
$$4(\tfrac{1}{3}\Gamma^3(\tfrac{1}{3}))^{\frac{1}{2}}\{\textstyle\int\int|x|^{\frac{3}{2}}f^2\int\int|y|^{\frac{3}{2}}f^2\int\int|xy|^{\frac{3}{2}}f^2\}^{\frac{1}{6}}$$

is an upper bound for $\int\int|f(x, y)|dxdy$.]

In addition, the author shows the relevance of the set K_S for Hölder's inequality and its generalizations. He also investigates the convergence of $\int_E P^{-1}dX$ when E is the set $x_i\geqq a>0$ $(i=1, \cdots, n)$; here the criterion is that K_S contains a point in the region $\alpha_\nu>1$ $(\nu=1, \cdots, n)$.

R. P. Boas, Jr. (Providence, R. I.).

Theory of Sets, Theory of Functions of Real Variables

Vera, Francisco. Theory of sets. Revista Acad. Colombiana Ci. Exact. Fis. Nat. **5**, 230–240 (1942). (Spanish) [MF 9097]

Albuquerque, J. The Cantor-Bendixson theorem. Gaz. Mat., Lisboa **5**, no. 20, 7–8 (1944). (Portuguese) [MF 12419]

Pi Calleja, Pedro. On Pincherle's lemma. Revista Union Mat. Argentina **10**, 15–18 (1944). (Spanish) [MF 11029]

This paper continues a discussion of a lemma of Pincherle which was initiated by J. Rey Pastor [Revista Union Mat. Argentina **9**, 29–35 (1943); these Rev. **5**, 1]. The author suggests a method different from that of Rey Pastor for correcting an error in Pincherle's lemma as originally stated. He concludes with a discussion of the applicability of this theorem and of various types of Borel covering theorems in the proof of the Cauchy-Goursat theorem.

J. V. Wehausen (Washington, D. C.).

Valentine, F. A. A Lipschitz condition preserving extension for a vector function. Amer. J. Math. **67**, 83–93 (1945). [MF 11924]

The author defines "property E" essentially as follows. In metric spaces $\mathfrak{M}$, $\mathfrak{M}'$ suppose that to each sphere S_i (center x_i, radius r_i) of a set M in $\mathfrak{M}$ there corresponds a sphere S_i' (center x_i', radius r_i') in $\mathfrak{M}'$; let M' be the set of such spheres S_i', and suppose that $r_i=r_i'$ and $\|x_i', x_j'\|'\leqq\|x_i, x_j\|$ for corresponding spheres. Then $\mathfrak{M}$ and $\mathfrak{M}'$ have the extensibility property E if under the preceding conditions $\prod_M S_i\neq0$ implies $\prod_{M'} S_i'\neq0$. The principal results of this paper are as follows. (i) Let $f(x)$ be a function which maps a set S in a metric space $\mathfrak{M}$ into a set S' in a metric space $\mathfrak{M}'$ so as to preserve the Lipschitz condition. A sufficient condition that $f(x)$ can be extended to any set $T\supset S$ so as to preserve the Lipschitz condition is that $\mathfrak{M}$ and $\mathfrak{M}'$ have property E. (The author states that this condition is also necessary.) (ii) The extension described in (i) holds when (1) $\mathfrak{M}=\mathfrak{M}'=n$-dimensional Euclidean space $\mathfrak{R}_n$; (2) $\mathfrak{M}=\mathfrak{M}'=$surface of an $(n+1)$-dimensional Euclidean sphere in $\mathfrak{R}_{n+1}$; (3) $\mathfrak{M}=\mathfrak{M}'=$a general Hilbert space. For cases (1) and (3) the extension can be defined so as to be contained in any closed convex set $L\supset S'$.

C. C. Torrance (Cleveland, Ohio).

Massera, José L. An example of a Jordan curve whose projections on three orthogonal planes fill areas. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadistica **1**, 95–98 (1944) = Bol. Fac. Ingen. Montevideo **2**, (Año 9), 669–672 (1944). (Spanish. French summary) [MF 11505]

An example is given of a "continuous curve" in three-dimensional Euclidean space which has zero three-dimensional measure but whose projections on three given orthogonal planes fill a hexagon in each plane.

J. V. Wehausen (Washington, D. C.).

Massera, José L. On differentiable functions. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadistica **1**, 71–93 (1944) = Bol. Fac. Ingen. Montevideo **2**, (Año 9), 647–668 (1944). (Spanish. French summary) [MF 11504]

This paper is concerned with functions which have a finite or infinite derivative at each point of a closed interval. In the first part, a variety of elementary theorems concerning derivatives are proved by using some simple properties of the difference quotient $p(a, b)=(f(a)-f(b))/(a-b)$. The second part of the paper is concerned chiefly with "Darboux functions": functions such that, for any two points x_1, x_2 in the interval of definition of $f(x)$, $f(x)$ takes on all values between $f(x_1)$ and $f(x_2)$ for values of x between x_1 and x_2. The most important theorems seem to be the following. (A) If $f(x)$ is a Darboux function in the closed interval $[a, b]$ and has a derivative at every interior point, $f'(x)$ has the properties of Rolle, Lagrange, and Darboux. (B) If the function $f(x)$ has a derivative at every point of $[a, b]$ and no discontinuities of the first kind, $f'(x)$ has the properties mentioned in (A).

J. V. Wehausen.

Haupt, Otto. Über Kontinua mit unvollständigen lokalen Halbsekantenmengen. J. Reine Angew. Math. 185, 231–240 (1943). [MF 12104]

Call a plane continuum C weakly (strongly) differentiable at a point p of C, if (for a given $\epsilon>0$) a neighborhood U of p exists such that $CU-p$ is contained in a union of disjoint open angular domains with vertex p each of which has opening less than π (ϵ). Let "nearly everywhere" mean "everywhere except at an at most countable number of points." Then, if C is nearly everywhere strongly differentiable, it has nearly everywhere exactly one tangent. If C is nearly everywhere weakly differentiable, it is a regular curve (in Menger's sense) and a hereditary arc sum.

H. Busemann (Chicago, Ill.).

Picone, Mauro. Sull'integrazione delle funzioni. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 3, 121–137 (1942) = Ist. Naz. Appl. Calcolo (2) no. 136. [MF 11722]

Picone, Mauro. Ancora sull'integrazione delle funzioni. Ist. Naz. Appl. Calcolo (2) no. 157, 11 pp. (1943). [MF 11768]

The author, inspired by an idea of L. Amerio [Boll. Un. Mat. Ital. (2) 4 (1942) = Ist. Naz. Appl. Calcolo (2) no. 143; these Rev. 6, 120], defines the Lebesgue integral on arbitrary sets in Euclidean n-space in the following way. Let $\int_C f dT$ be already defined for closed, bounded sets C and for continuous integrands f. [Cf. M. Picone, Lezioni di Analisi Infinitesimale, Catania, 1923, p. 377.] If now E is an arbitrary set and f an arbitrary nonnegative function, he defines

$$\int_E f dT = \lim_{C\to E} \int_C f dT,$$

where C are those closed bounded subsets of E on which f is continuous. (Such subsets C exist; for instance, the empty set or finite sets; and the limit has meaning, since the integral for nonnegative f is a monotone increasing set function.) If $f=1$, this gives the inner Lebesgue measure of E. For an f with varying sign one can operate analogously. He calls f "summable" on E if $\int_E |f| dT$ has a finite value; in this case $\int_E f dT$ exists and is finite also. Furthermore, the author discusses an extension to Lebesgue-Stieltjes integrals.

In the second note the author gives some more explanations answering a criticism of L. Tonelli [Ann. Scuola Norm. Super. Pisa (2) 11, 235–240 (1942)]. *A. Rosenthal.*

Jeffery, R. L., and Miller, D. S. Convergence factors for generalized integrals. Duke Math. J. 12, 127–142 (1945). [MF 12075]

This paper generalizes results obtained by Burkill [Proc. London Math. Soc. (2) 39, 541–552 (1935)] on Cesàro-Perron integration by replacing the Cesàro mean

$$C_r(F, x, x+h) = (r/h^r) \int_x^{x+h} (x+h-t)^{r-1} F(t) dt$$

by

$$G(F, x, x+h) = (1/\theta(x, h)) \int_x^{x+h} v(x, x+h, t) F(t) dt,$$

where $v(x, x+h, t) = v(x+h, x, t)$ is positive for almost all t and Lebesgue integrable in $(x, x+h)$,

$$\theta(x, h) = \int_x^{x+h} v(x, x+h, t) dt,$$

and $F(t)$, defined on (a, b), is such that the integral used in $G(F, x, x+h)$ exists in a suitably defined sense. By defining upper and lower G-derivatives in terms of the ratio

$$[G(F, x, x+h) - F(x)]\lambda/h,$$

where

$$1/\lambda = \lim_{h\to 0} \int_x^{x+h} v(x, x+h, t)(t-x) dt/h\theta(x, h),$$

it is possible to define upper and lower functions for a function $f(x)$ and consequently a generalized Perron integral. There follow the usual properties of a Perron type of integral including a theorem of considerable scope on integration by parts. *T. H. Hildebrandt.*

Choquet, Gustave. Primitive d'une fonction par rapport à une fonction à variation non bornée. C. R. Acad. Sci. Paris 218, 495–497 (1944). [MF 12119]

This note outlines a method for calculating a function $F(x)$ when its derivative $g(x)$ with respect to a function $\alpha(x)$ is known. The function $\alpha(x)$ is not necessarily of bounded variation. This problem was proposed by Lebesgue [Leçons sur l'intégration, second ed., Gauthier-Villars, Paris, 1928, p. 313]. Results are obtained for functions on general compact spaces. Let $F(M)$ and $\alpha(M)$ be two numerical-valued functions defined on a topological space E. If $[F(M')-F(M)]/[\alpha(M')-\alpha(M)]\to g(M)$ as $M'\to M$ in such a way that the numerator and denominator of this ratio are not simultaneously zero, $g(M)$ is the derivative of F with respect to α. If both terms of the ratio are zero for all points M' in a neighborhood of M, the derivative is indeterminate at M. Let $g(M)$ be defined or indeterminate. Let F be the points where $g(M)$ is defined, and suppose that $g(M)$ is finite except on a set $I\subset F$, not contain a perfect set. Under these hypotheses all primitive functions of $g(M)$ with respect to $\alpha(M)$ are of the form $F(M)$+constant. *R. L. Jeffery* (Kingston, Ont.).

Rado, Tibor. On surface area. Proc. Nat. Acad. Sci. U.S.A. 31, 102–106 (1945). [MF 12043]

The writer begins by reviewing the concepts of the essential multiplicity function $\kappa(\xi, \eta; T, D)$, the generalized Jacobian, and the essential absolute continuity (eAC) and essential bounded variation (eBV) of a continuous transformation $\xi=\xi(u, v)$, $\eta=\eta(u, v)$ defined on a bounded domain D. For surfaces S of the type of the 2-cell, he defines a new "lower area" $a(S)$ in terms of the essential multiplicity functions $\kappa(x_i, x_j; T_{ij}, D)$ of the mappings $T_{ij}: x_i=x_i(u, v)$, $x_j=x_j(u, v)$ on subregions D.

The writer then states the following results. (1) Suppose that the Lebesgue area $A(S)$ is finite and that the partial derivatives of a set of representing functions exist almost everywhere. Then the classical area integral I exists in the Lebesgue sense with $I\leqq A(S)$, the equality holding if and only if all the mappings T_{ij} are eAC. (2) If $A(S)<\infty$, then any representation is eBV (that is, all the T_{ij} are) and $I_e\leqq A(S)$, I_e being the classical area integral with generalized Jacobians replacing ordinary ones; the equality holds if and only if all the T_{ij} are eAC. (3) Let $A(S)<\infty$, let the 2-cell Q be subdivided into simply connected subregions $R_1, \cdots, R_m$ with boundaries $C_1, \cdots, C_m$, and let S_i be the surface corresponding to the 2-cell R_i+C_i, $i=1, \cdots, m$. If the projections of the images C_i^* of each C_i on each coordinate plane are all of plane measure zero, then $A(S)=A(S_1)+\cdots+A(S_m)$. (4) The lower area $a(S)$ is never greater than $A(S)$. If either $A(S)<\infty$ or $a(S)=0$, then $a(S)=A(S)$. If S admits a representation of the form $z=f(x, y)$, then $a(S)=A(S)$. (5) Let S be a surface, let p be a plane through

the origin O, let OP be perpendicular to p, P being on the unit sphere, let ξ, η be Cartesian coordinates in p, let $\kappa_p(\xi, \eta; Q^0)$ denote the essential multiplicity function for the projection of S on p, and let $\alpha(P)$ be the integral of $\kappa_p(\xi, \eta; Q^0)$ over p. If $A(S) < \infty$ then $\alpha(P)$ is summable and its mean value over the sphere is equal to $\frac{1}{4}A(S)$.

C. B. Morrey, Jr. (Aberdeen, Md.).

Loomis, Lynn H. Abstract congruence and the uniqueness of Haar measure. Ann. of Math. (2) **46**, 348–355 (1945). [MF 12407]

Let M be a locally compact metric space such that if two compact spheres have the same radius and one can be covered by N open spheres of radius x then so can the other. It is shown that there exists one and (up to a constant factor) only one countably additive measure in M, defined for every Borel set which can be covered by countably many compact sets, and such that compact spheres of equal radius have equal positive finite measure. This result gives a new proof of the familiar existence and uniqueness theorems for Haar's measure in the case of metric groups, and the extension to nonmetrizable locally compact groups and certain uniform structures is indicated. Apart from the fact that it establishes uniqueness assuming only a very weak congruence relation, the special feature of the present proof is that both existence and uniqueness are proved simultaneously without using the axiom of choice. This has been done hitherto only in the case of groups, and by a different method. The present method follows the older proofs of Haar and Banach in defining the measure directly through the limits of covering ratios, rather than defining it indirectly through an invariant integral.

J. C. Oxtoby (Bryn Mawr, Pa.).

Blumberg, Henry. Arbitrary point transformations. Duke Math. J. **11**, 671–685 (1944). [MF 11570]

With x a point of Euclidean m-space S_m and y a point of Euclidean n-space S_n, assume only that $\tau: y=f(x)$ is one-valued and defined for every x in S_m. Let S denote the $(m+n)$-space of (x, y). Let I denote the set of points $[x, f(x)]$ as x ranges over S_m.

An open oriented cylinder C_{br}, where b is in S_n and r is a positive real number, is the set of points of S such that the distance from y to b is less than r. If A is a set in S, by $X(A)$ is meant the set of points of S_m which are x-coordinates of points of A. A salient point of τ is a point $\zeta:(\xi, \eta)$ of S such that, for every open oriented cylinder C containing ζ, the set $X(IC)$ has positive upper density at ξ (in the sense of exterior Lebesgue measure). One of the theorems proved is that for every point transformation τ there exists a dense subset D of S_m such that τ is continuous on D with respect to D. In the proof the author uses the definition that a point z of a Euclidean space Σ is "inexhaustibly approached" by a subset M of Σ if, for every neighborhood N of z, MN is not the sum of $\aleph_0$ nondense sets.

Now let x be restricted to be in $U: 0<x_i<1$ $(i=1, \cdots, m)$. A finite oriented prism is the set of points of S for which x is in U and $a_j<y_j<b_j$ $(j=1, \cdots, n)$. If X is a subset of U, the point ξ of U is a metric limit point of X in the sense of exterior measure if the upper density of X is positive at ξ. A subset A of S, possibly containing "infinite points," is projectionally m-dense-in-itself if, for every (finite) oriented prism P, the set $X(PA)$ is metrically dense-in-itself, and, for every "infinite point (ξ, ∞) of A and every open infinite oriented prism P," the point ξ is a metrical limit point of $X(PA)$. A subset A of S, possibly containing infinite points, is projectionally m-closed if, for every closed oriented prism P, whether finite or infinite, $X(PA)$ is metrically closed; it is projectionally m-perfect if it is both projectionally m-dense-in-itself and projectionally m-closed. The principal theorem follows. Necessary and sufficient conditions that a subset T of S be the set of salient points of a (one-valued finite) point transformation τ are that (a) the product of T and any linear space $x=\xi$ is m-closed and not empty; (b) T is projectionally m-perfect.

A. B. Brown.

Theory of Functions of Complex Variables

Ríos, S. Scheme for a simplified exposition of the theory of analytic functions. Revista Mat. Hisp.-Amer. (4) **3**, 310–311 (1943). (Spanish) [MF 12157]

The author outlines an exposition in which a function is defined to be analytic in a simply connected region if it is the limit of a sequence of polynomials converging uniformly in every closed subregion. After Cauchy's integral theorem has been proved for polynomials, all standard results follow easily.

R. P. Boas, Jr. (Providence, R. I.).

Vermes, P. Geometric representation of analytic functions. Math. Gaz. **29**, 4–9 (1945). [MF 12203]

Carlson, Fritz. Quelques inégalités concernant les fonctions analytiques. Ark. Mat. Astr. Fys. **29B**, no. 11, 6 pp. (1943). [MF 12023]

Let $f(z)$ be analytic in the unit circle C and let L be an arbitrary curve in the interior of C. Then

$$\int_L |f(z)dz| < (1/\pi)\int_C |f(z)dz| \cdot V(z),$$

where $V(z)$ denotes the sum of angles under which the elements of the curve L appear from a point z on C. For instance, for a chord L of C we have $V(z)<\pi$; in particular, for a diameter $V(z)=\pi/2$. For a convex curve L we have $V(z)<\pi$. These special cases lead to known inequalities [Fejér and F. Riesz, Math. Z. **11**, 305–314 (1921); Gabriel, Proc. London Math. Soc. (2) **28**, 121–127 (1928)]. An analogous inequality is derived for the half-plane. Moreover, in all the inequalities $f(z)$ can be replaced by $(f(z))^k$.

G. Szegö (Stanford University, Calif.).

Golomb, Michael. The convergence of sequences of Hadamard determinants. Duke Math. J. **11**, 759–777 (1944). [MF 11578]

Let $f(z)=\sum c_n z^n$ be regular in the closed circle C except for the p poles $z_1, \cdots, z_p$ inside and the q poles $z_{p+1}, \cdots, z_{p+q}$ on the circle. Let

$$d_n^{(m)}=\begin{vmatrix} c_{n+1} & \cdots & c_{n+m} \\ \cdot & \cdot & \cdot \\ c_{n+m} & \cdots & c_{n+2m-1} \end{vmatrix}.$$

Cases where $\lim_{n\to\infty} d_n^{(m)}/d_{n+1}^{m}$ exists have been considered by Hadamard and by Golomb [Bull. Amer. Math. Soc. **49**, 581–592 (1943); these Rev. **5**, 49]. In this paper it is shown that the convergence does not in general depend on the relative position of the poles but only on their multiplicities, and a complete characterization of the numbers m for which there is convergence is obtained.

E. C. Titchmarsh (Oxford).

Ríos, S. A demonstration of a theorem of Ostrowski. Revista Mat. Hisp.-Amer. (4) 3, 361–364 (1943). (Spanish) [MF 12178]

The theorem states that, if $f(z)=\sum a_n z^n$ ($|z|<1$), with $a_n=0$ for $n_k<n\leqq n_k'$, where $n_k'/n_k\rightarrow\infty$, then the sequence of partial sums of order n_k converges uniformly to $f(z)$ in a neighborhood of every regular point [Ostrowski, J. London Math. Soc. 1, 251–263 (1926)]. Ostrowski's proof used the three-regions theorem. The author avoids this by using the function which maps a region Δ on the unit circle, where Δ connects the origin with a regular point of $f(z)$ outside $|z|\leqq 1$. *R. P. Boas, Jr.* (Providence, R. I.).

Salem, R. Power series with integral coefficients. Duke Math. J. 12, 153–172 (1945). [MF 12077]

Let $\theta=\theta_1$ be an algebraic integer greater than 1, of arbitrary degree n, with conjugates $\theta_2, \cdots, \theta_n$; the number θ lies in the set P or T if $|\theta_k|<1$ or $|\theta_k|\leqq 1$, $k=2, \cdots, n$, respectively. It is proved that each number of P is a limit point of numbers of T, on both sides, and that P is closed; the second fact had previously been established by the author [Duke Math. J. 11, 103–108 (1944); these Rev. 5, 254]. By a theorem of C. Pisot [Ann. Scuola Norm. Super. Pisa (2) 7, 205–248 (1938)], a number $x>1$ belongs to P if and only if $\sum_{l=0}^{\infty}\sin^2(\pi x^l)$ converges. The author obtains a corresponding criterion for T; furthermore, as a consequence of Pisot's results, he proves that a nonrational power series with integral rational coefficients, converging in the unit circle, assumes values arbitrarily near to any number, and he generalizes this theorem to the case of a function which is only meromorphic in the unit circle. *C. L. Siegel* (Princeton N. J.).

Schaeffer, A. C., and Spencer, D. C. The coefficients of schlicht functions. II. Duke Math. J. 12, 107–125 (1945). [MF 12074]

[Part I appeared in the same J. 10, 611–635 (1943); these Rev. 5, 175.] Let $G(z)$, with $G(0)=0$, $G'(0)=1$, be regular and schlicht for $|z|<1$. For a sufficiently small positive γ and for $0<a<1$ the function $g(z)=\gamma a^{-1}G(az)$ maps $|z|=1$ into a simple analytic Jordan curve lying within the unit circle. The authors derive a differential equation somewhat similar to that of K. Löwner [Math. Ann. 89, 103–121 (1923)]. They show there is a function

$$g(z,t)=\gamma e^t(z+a_2(t)z^2+\cdots)$$

defined for $|z|<1$ and $0\leqq t\leqq T=\log\gamma^{-1}$, with $g(z,T)=z$, $g(z,0)=g(z)$, satisfying the differential equation

$$g_t(z,t)=zp(z,t)g_z(z,t),$$

where

$$p(z,t)=1+2\sum_{\nu=1}^{\infty}c_\nu(t)z^\nu$$

is regular and has a positive real part for $|z|<1$. Löwner's classical recursion formula for the coefficients $a_n(t)$ is obtained. The new version simplifies the proof that the third coefficient of a function schlicht and normalized for $|z|<1$ is bounded by 3. The authors show that the a_n obtained from their recursion formula when the c's associated with $p(z,t)$ are real and continuous lie everywhere dense in the coefficient space of schlicht functions all of whose coefficients are real.

For functions bounded by e^T the maximum value of $|a_3|$ is found. For $T<1$ it is $1-e^{-2T}$; for $T\geqq 1$ it is $1-e^{-2T}+2(1-x_0)^2e^{-2x_0}$, where $x_0e^{x_0}=e^{-T}$ and $0<x_0\leqq 1$. An interesting description of the map of $|z|=1$ by the extremal function maximizing $|a_3|$ is given.

Let R be an open simply-connected domain containing the origin, of mapping radius unity, and let $f(z)=z+\sum_2^{\infty}a_nz^n$ be regular and schlicht in R. If R is the unit circle it has been conjectured that $|a_n|\leqq n$ for all n. The inequality is known to be true when the coefficients are all real. The bounds for $n=2$ and 3 are attained when the coefficients are all real. The authors raise the question: for all regions R is the maximum $|a_3|$ attained by a function $f(z)$ all of whose coefficients are real (this being the situation when max $|a_2|$ is attained)? They show that the answer is negative.

Using the variational method described in part I, the authors give a new proof for Löwner's bound for the coefficients of the power series of the function $\zeta=F(z)$ inverse to the function $z=f(\zeta)$, $f(0)=0$, $f'(0)=1$, regular and schlicht for $|\zeta|<1$.

This method is also applied to special polynomials in the coefficients $a_2, a_3, \cdots$ of a function $f(\zeta)$ regular and schlicht in the unit circle. Let

$$f(\zeta)/f'(\zeta)=2\sum_{n=1}^{\infty}\alpha_n\zeta^n,\quad f(\zeta)=\zeta+\sum_{n=2}^{\infty}a_n\zeta^n.$$

The authors point out that there is much "evidence" that $|\alpha_4|\leqq 1$; (i) it is true for star-like functions f; (ii) $|\alpha_2|\leqq 1$, $|\alpha_3|\leqq 1$; (iii) the differential equation satisfied by $f(\zeta)$ maximizing $|\alpha_4|$ is also satisfied by $\zeta(1+\zeta^k)^{-2/k}$, $k=1, 3$. Nevertheless it is shown that $|\alpha_4|$ can exceed 1.

In the last section the authors simplify the proof of the fundamental differential equation satisfied by $f(\zeta)$ maximizing $|a_n|$. *M. S. Robertson* (New Brunswick, N. J.).

Chandrasekharan, K. On the canonical expression for a meromorphic function of finite order. J. Madras Univ. Sect. B. 15, 11–17 (1943). [MF 12317]

The author extends his proof of Hadamard's factorization theorem [J. Indian Math. Soc. (N.S.) 5, 128–132 (1941); these Rev. 3, 201] to obtain a proof of R. Nevanlinna's factorization theorem for a meromorphic function of finite order [Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 213]. *R. P. Boas, Jr.*

Shah, S. M. A note on the maximum modulus. J. Univ. Bombay (N.S.) 13, part 3, 3 pp. (1944). [MF 12061]

For an entire function $f(z)$ let $M(r)$ and $M_1(r)$ denote the maxima of $|f(z)|$ and $|f'(z)|$ for $|z|\leqq r$. If $f(z)$ is of order ρ then

$$(*)\qquad \limsup_{r\to\infty}\frac{\log\{rM_1(r)/M(r)\}}{\log r}=\rho;$$

this was apparently known before only for functions with positive coefficients. For functions with positive coefficients, the author shows that replacing lim sup by lim inf in (*) replaces ρ by the lower order λ. He constructs, for any positive increasing $\varphi(x)$, an entire function for which $\limsup M_1(r)/\varphi(M(r))=\infty$; the possibility of such a construction was stated by Vijayaraghavan [J. London Math. Soc. 10, 116–117 (1935)]. *R. P. Boas, Jr.*

Dufresnoy, Jacques. Remarques sur les fonctions méromorphes dans le voisinage d'un point singulier essentiel isolé. Bull. Soc. Math. France 70, 40–45 (1942). [MF 11888]

The following theorem, which extends earlier results of Ahlfors [Acta Math. 65, 157–194 (1935)], is demonstrated.

Let there be given q ($\geqq 3$) simply-connected closed disjoint domains D_i on the Riemann sphere (some of which may reduce to points) and let there be associated with each D_i a positive integer μ_i greater than one such that

$$\sum_i (1-1/\mu_i) > 2.$$

Furthermore, let $w=f(z)$ denote a function which is single-valued and meromorphic for $r_0<|z|<\infty$ and let Σ denote the Riemannian image of $r_0<|z|<\infty$ with respect to $w=f(z)$. If $z=\infty$ is an essential singularity of $f(z)$, then on at least one of the D_i, say D_k, the surface Σ has an infinite number of discs with less than μ_k sheets. Several proofs are given of this theorem. One proof is based upon appraisals relating length and area. Another reduces the theorem to the corresponding one for functions which are meromorphic in the finite z-plane. Related results are indicated.

M. H. Heins (Arlington, Va.).

Warschawski, S. E. On Theodorsen's method of conformal mapping of nearly circular regions. Quart. Appl. Math. 3, 12–28 (1945). [MF 12082]

A nearly circular contour C is defined as a curve which can be expressed in polar coordinates by $\rho=\rho(\theta)$, $0\leqq\theta\leqq 2\pi$, where ρ is absolutely continuous and $a(1+\epsilon)^{-1}\leqq\rho(\theta)\leqq a(1+\epsilon)$, $|\rho'(\theta)/\rho(\theta)|\leqq\epsilon$, $a>0$, $0<\epsilon<1$. Theodorsen developed a method of computing the function which maps the exterior of C conformally on the exterior of the unit circle. This function was obtained as the solution by successive approximations of a nonlinear integral equation; however, proofs of convergence and estimates of rapidity of convergence were not carried out. In the present paper, these proofs and estimates are supplied.

The problem is normalized by considering the function $w=f(z)$ which maps $|z|<1$ on the interior of C, with $f(0)=0$ and $f'(0)>0$; this can be reduced to Theodorsen's problem by inversion. The single-valued function

$$\log [f(z)/z] = \log |f(z)/z| + i \arg [f(z)/z]$$

is considered. If $\arg f(e^{i\phi})/e^{i\phi}$ is written as $\theta(\phi)-\phi$, then $F(e^{i\phi})=\log\rho[\theta(\phi)]+i[\theta(\phi)-\phi]$. From the integral identity for analytic functions F in the unit circle,

$$F(e^{i\phi})-F(0)=\frac{1}{2\pi i}\int F(\zeta)\frac{e^{i\phi}+\zeta}{e^{i\phi}-\zeta}\frac{d\zeta}{\zeta},$$

the integral equation

$$\theta(\phi)-\phi=-(1/2\pi)\int \{\log\rho[\theta(\phi+t)] - \log\rho[\theta(\phi-t)]\}\cot\tfrac{1}{2}t\,dt$$

results. This equation is solved by successive approximations starting with $\theta_0(\phi)=\phi$. Estimates of the following sort are obtained:

$$|\theta_n(\phi)-\theta(\phi)|\leqq 2(\pi^2/(1-\epsilon^2))^{\frac{1}{2}}\epsilon^{\frac{1}{2}(n+2)},$$

which lead to corresponding estimates for the mapping functions themselves. By making more restrictive assumptions on C, estimates for the derivatives of the mapping functions are obtained, as well as more rapidly converging estimates for the mapping function. Interesting results are also obtained on conditions sufficient to insure that the approximating curves are star shaped.

J. W. Green.

Denjoy, Arnaud. Les continus cycliques et la représentation conforme. Bull. Soc. Math. France 70, 97–124 (1942). [MF 11891]

The author defines a "cyclic continuum" as the frontier of an open connected set which cannot be decomposed into disjoint continua of which an infinite number would have a diameter greater than an arbitrary, but fixed, positive number. The paper is concerned with the proof of the following theorem. If $Z=\sum a_n z^n$ represents the region R (of a Riemann surface) conformally on $|z|<1$, and if R has a finite area, then a necessary and sufficient condition for the uniform convergence of $\sum a_n e^{in\theta}$ is that the frontier of R should be a cyclic continuum.

František Wolf.

Ferrand, Jacqueline. Étude de la correspondence entre les frontières dans la représentation conforme. Bull. Soc. Math. France 70, 143–174 (1942). [MF 11893]

A detailed study is made of the boundary behavior of a function $z=\varphi(\zeta)$ which defines a (1, 1) directly conformal map of a simply-connected region Δ with boundary Γ (not reducing to a point) onto $|z|<1$. It is indicated that the present study admits an extension to the case of (1,1) quasi-conformal maps [Ahlfors, Acta Math. 65, 157–194 (1935), especially p. 185]. The arguments employed make considerable use of appraisals on length and area derived with the aid of Schwarz's inequality. The following is typical of the theorems proved. If $\zeta\epsilon\Delta$ tends to a prime end Ω of Γ, then $\varphi(\zeta)$ tends to a uniquely determined point a of $|z|=1$ and $|\varphi(\zeta)-a|<2\pi|\log R|^{-\frac{1}{2}}$ if ζ and Ω can be separated from a given point ζ_0 of Δ by the same circular cut of radius R. The harmonic measure of certain sets of prime ends of Γ is studied.

M. H. Heins.

Ferrand, Jacqueline. Étude de la représentation conforme au voisinage de la frontière. Ann. Sci. École Norm. Sup. (3) 59, 43–106 (1942). [MF 11879]

This memoir is concerned with the boundary behavior of a function $\zeta=f(z)$ which maps $|z|<1$ directly conformally and (1,1) onto a simply-connected region Δ with boundary Γ in the ζ-plane. Considerable use is made of appraisals of length and area. The major results of this study have been announced in a series of notes [C. R. Acad. Sci. Paris **212**, 977–980 (1941); **213**, 638–640 (1941); **214**, 50–52, 250–253 (1942); **215**, 254–255 (1942); these Rev. **5**, 37, 115; **4**, 138, 139; **5**, 94].

The first section of the paper finds its genesis in the following theorem of Denjoy [C. R. Acad. Sci. Paris **212**, 1071–1074 (1941); **213**, 15–17 (1941); these Rev. **5**, 115, 116]. If $f(z)$ is analytic and univalent in $|z|<1$, then, for almost all points a on the unit circle, one has $f'(z)=o((z-a)^{-\frac{1}{2}})$ as z tends to a in a sector of angular opening less than π, the sector having a as its vertex and the ray Oa as bisector. This result is established under much weaker hypotheses in chapter I with the aid of the lemma of H. Cartan and Ahlfors. The condition of univalence is considerably relaxed. Chapter II applies the results of chapter I to univalent functions. Questions of accessibility are treated and results of Carathéodory and Lindelöf are recast in metric terms. The set of limiting values of $f(z)$ as z tends to a boundary point in a specified manner is studied as well as the set of points on $|z|=1$ for which the set of limiting values does not reduce to a point.

Chapter III is concerned with the behavior of the mapping function in the neighborhood of an antecedent a of an accessible boundary point α of Γ. Here use is made of the methods of Ostrowski [Prace Mat.-Fiz. 44, 371–471 (1937)]. Sufficient conditions of a geometric character are given which guarantee that $f(z)$ has the unique limit α as z tends to a on a curve having an assigned order of contact with the unit circle at a. In chapter IV the work of Ahlfors on

the existence of an angular derivative [Acta Soc. Sci. Fennicae. Nova Ser. A. 1, no. 9 (1930)] is extended and sufficient conditions of a broad nature are given. The advances made are illustrated by a set of examples.

M. H. Heins (Arlington, Va.).

Petersson, Hans. Ein Summationsverfahren für die Poincaréschen Reihen von der Dimension —2 zu den hyperbolischen Fixpunktepaaren. Math. Z. 49, 441–496 (1944). [MF 11993]

Let Γ be a Fuchsian group of the first kind with the matrix elements

$$L=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

where α, β, γ, δ are real numbers and $\alpha\delta-\beta\gamma=1$; let $z=x+iy$, $\zeta=\xi+i\eta$, $y>0$, $\eta<0$; then the Poincaré series

$$f_r=f_r(z,\zeta)=\sum_L \{(\alpha z+\beta)-\zeta(\gamma z+\delta)\}^{-r}$$

is an automorphic form of the variable z with weight r, provided r is an even integer greater than 2. If Γ contains the special parabolic substitution $z\to z+1$, then $f_r(z,\zeta)=f_r(z,\zeta+1)$ and the Fourier expansion

$$f_r=\sum_{k=-\infty}^{\infty}\phi_k e^{2\pi ik\zeta}$$

exists, where the coefficients ϕ_k again are automorphic forms. In former publications the author generalized this expansion in two directions. He considered automorphic forms with arbitrary real weight $r>2$, having an arbitrary multiplier system of absolute value 1, and he replaced the special substitution $z\to z+1$ by an arbitrary parabolic or hyperbolic substitution in Γ; the different expansions of a given Poincaré series implied a number of remarkable relations among automorphic forms. Now he investigates the same problems for the limiting case $r=2$, under the assumption that Γ is a congruence subgroup of the modular group; the proofs become much more delicate, since the series $f_2(z,\zeta)$ is not absolutely convergent. As a possible application of his new results he sketches a method of constructing the period matrix of the Abelian integrals of the first kind for the corresponding algebraic function fields.

C. L. Siegel (Princeton, N. J.).

Martin, W. T. Functions of several complex variables. Amer. Math. Monthly 52, 17–27 (1945). [MF 11907]

In diesem Vortrag gibt der Verfasser einen Überblick über einige für die Theorie der Funktionen mehrerer komplexen Veränderlichen typische Ergebnisse, für die es in der klassischen Funktionentheorie keine Analogie gibt oder die auf diese Theorie angewandt falsch sind. Im besonderen wird auf den Hartogs'schen Satz eingegangen, nach dem eine auf dem zusammenhängenden Rande eines beschränkten Bereiches des R_{2n} analytische Funktion $f(z_1,\cdots,z_n)$ sich ins ganze Innere des Bereiches hinein analytisch fortsetzen lässt. Ferner werden die Probleme von Cousin und Poincaré behandelt, d.h. die Versuche, den Mittag-Leffler'schen und den Weierstrass'schen Satz über die Existenz von zu gegebenen Pol- bzw. Nullstellen gehörigen meromorphen bzw. analytischen Funktionen auf die Theorie der Funktionen mehrerer komplexen Veränderlichen zu übertragen, und das Problem der Quotientendarstellung einer meromorphen Funktion. *P. Thullen.*

Wachs, S. Sur les transformations pseudo-conformes admettant un point frontière invariant. J. Math. Pures Appl. (9) 22, 25–54 (1943). [MF 12249]

Using the notion of the kernel of a domain in the space of two complex variables, the notion of domains of comparison, the classification of boundary points and other concepts developed by S. Bergman, the author considers analytic (pseudo-conformal) transformations of a domain B into a domain $G\subset B$ which leave invariant a given boundary point. The following theorem is typical of the results obtained. Let B be a suitably restricted domain having the domain $A=E[(z_1-r_1)>r_2,\ (z_2)<R_2]$ as an exterior domain of comparison and $I=E[(z_1-r_1)<r_1,\ (z_2)<R_1]$ as an inner domain of comparison, let Q be a boundary point of the second order and third kind, and let T be an analytic transformation such that $TB=G\subset B$. If there exists a denumerable sequence of points $\{P_n\}=\{z_1^{(n)}, z_2^{(n)}\}$ such that $P_n\to Q$ and $TP_n\to Q$ in such a manner that the expression

$$L^{(n)}=F(z_1^{*(n)})/F(w_1^{*(n)})\to\Gamma,\qquad 0<\Gamma<\infty,$$

then for every point $\{z_1, z_2\}$ sufficiently near Q the inequality

$$\Gamma F(z_1^*)/|z_1^*|^2\geqq F(w_1^*)/|w_1^*|^2$$

holds. Here

$$F(u)=u+\bar u-u\bar u,\quad z_1^*=z_1/r_1\quad\text{and}\quad w_1^*=w_1/(w_1+r_2).$$

Two other results are obtained for mappings of this sort and three similar results are obtained for transformations leaving invariant a point of the fourth order.

W. T. Martin (Syracuse, N. Y.).

Nef, Walter. Die unwesentlichen Singularitäten der regulären Funktionen einer Quaternionenvariabeln. Comment. Math. Helv. 16, 284–304 (1944).

The author proves a conjecture of Fueter on nonessential singularities of regular functions of a quaternion variable. A closed set $\mathfrak{M}$ of points of quaternion space is called nonessentially singular for a right-regular function $f(z)$ if (1) every point of $\mathfrak{M}$ is a singular point of $f(z)$, (2) there exist a constant M and a positive integer N such that

$$|f(z)|<M\rho^{-N}/J(\rho),$$

where ρ is the distance of the point z from the set $\mathfrak{M}$ and $J(\rho)$ is the lower bound of the surface area of all closed orientable hypersurfaces with continuous normal fields which contain $\mathfrak{M}$ in their interiors and whose points have a distance at least ρ from $\mathfrak{M}$. The smallest number which has the property of N is called the order of the nonessential singular set. The first main theorem is theorem 10: If $f(z)$ is right-regular in the interior of the closed hypersurface R with the exception of the points of a singular set $\mathfrak{M}$ which is a nonessential set of order N, then for all z which lie in the interior of R and which can be joined with R by a continuous curve which has no points in common with $\mathfrak{M}$ the following relation holds:

$$f(z)=\varphi(z)+\sum_{n=0}^{N}\sum_{n=n_1+n_2+n_3}\int_{\mathfrak{M}}d[\Delta_{n_1n_2n_3}(\mu)]\cdot q_{n_1n_2n_3}(z-c).$$

Here $\varphi(z)$ is a right-regular function in the interior of R; in particular, therefore, on $\mathfrak{M}$. In the first part of the paper the author develops the necessary tools relating to Stieltjes integrals on compact metric surfaces which he uses later. In the second part of the paper he obtains several results relating to the representation of regular functions in the neighborhood of singular point sets. The final result ob-

tained in this part is theorem 10 quoted above. In the third part of the paper he studies right-regular functions which have nonessential singularities in the finite portion of the space. His second main theorem gives a representation involving sums of Stieltjes integrals taken over the singular sets for a right-regular function having only isolated nonessential singular manifolds in the finite portion of the space. *W. T. Martin* (Syracuse, N. Y.).

Balseiro, Jose A. Elements of the theory of functions of an antoidal tricomplex variable. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 180, Vol. 3, num. 4. Serie segunda, 14, Contribuciones, 413–442 (1944). (Spanish) [MF 12192]

The author's "antoidal tricomplex numbers" are three-component numbers with a (commutative) multiplication in which $(a_1, a_2, a_3)\cdot(b_1, b_2, b_3)$ has components

$$(a_1b_1-a_2b_3-a_3b_2),\ (a_1b_2+a_2b_1-a_3b_3),\ (a_1b_3+a_3b_1+a_2b_2).$$

Every such number can also be written in the form (u, z), where u is a real number and z is an ordinary complex number, and $(u_1, z_1)\cdot(u_2, z_2)=(u_1u_2, z_1z_2)$. Using a natural definition of limit, the author discusses derivatives, series, etc., for antoidal functions of an antoidal variable. It turns out that a differentiable function is necessarily of the form $(f(u), g(z))$, where $f(u)$ is a (real) differentiable function of the "real part" u and $g(z)$ is an analytic function of the "complex part" z. The author also discusses geometric interpretations of antoidal numbers. *R. P. Boas, Jr.*

Theory of Series

Stenström, V. Remarque sur une règle de convergence uniforme. Ark. Mat. Astr. Fys. **30B**, no. 10, 2 pp. (1944). [MF 12128]

Example of a series $\sum u_n(x)$ of positive terms, uniformly convergent in $0\leqq x\leqq 1$, but such that there is no convergent series $\sum M_n$ with $u_n(x)\leqq M_n$. *R. P. Boas, Jr.*

Albuquerque, J. A theory of double series. Gaz. Mat., Lisboa **5**, no. 21, 1–4 (1944). (Portuguese) [MF 12422]

Allen, H. S. Projective convergence and limit in sequence spaces. Proc. London Math. Soc. (2) **48**, 310–338 (1945). [MF 12055]

This paper extends investigations of Köthe and Toeplitz [J. Reine Angew. Math. 171, 193–226 (1934); the author credits many of his definitions and results to lectures by P. Dienes]. For $n=1, 2, \cdots$, let $x^{(n)}\equiv(x_1^{(n)}, x_2^{(n)}, \cdots)$ denote a sequence of numbers. The sequence $\{x^{(n)}\}$ of sequences is coordinate convergent to the sequence $x\equiv(x_1, x_2, \cdots)$ if $\lim_{n\to\infty}x_k^{(n)}=x_k$ for each k. If a sequence $\{x^{(n)}\}$ in a linear manifold α of sequences is such that

$$(1)\qquad \lim_{n\to\infty}\sum_{k=1}^{\infty}x_k^{(n)}u_k=\sum_{k=1}^{\infty}x_ku_k$$

for each u in a linear manifold β of sequences, and if the series in (1) converges absolutely, then the sequence $\{x^{(n)}\}$ is projective convergent to x in α relative to β or $\alpha\beta$-convergent to x. Many associated definitions, too numerous to present here, are given. Many theorems relate $\alpha\beta$-convergence and coordinate convergence to each other and to other types of convergence. *R. P. Agnew.*

Neder, Ludwig. Über den Zusammenhang zweier Sätze von Lebesgue und Toeplitz. Math. Z. **49**, 576–578 (1944). [MF 11988]

It is asserted [erroneously] that the Silverman-Toeplitz theorem giving necessary and sufficient conditions for regularity of matrix methods of summability of sequences is an immediate consequence of a theorem of Lebesgue on singular integrals. Lebesgue's theorem gives necessary and sufficient conditions that for a kernel $K(s, t)$, assumed to be integrable over $0\leqq t\leqq a$ for each s,

$$(1)\qquad \lim_{s\to\infty}\int_0^a K(s, t)f(t)dt=\lim_{x\to 0}f(x)$$

for each function $f(x)$ having no discontinuities other than those of the first kind in the interval $0\leqq x\leqq a$. The author's assertion is false, and his indication of proof is illusory, because Lebesgue's theorem does not imply that his conditions are necessary in order that (1) may hold for each step function $f(t)$ having jumps at preassigned points. [For an exposition of the relation between theorems involving kernel transformations and theorems involving matrix transformations, see Agnew, Bull. Amer. Math. Soc. **45**, 689–730 (1939), in particular, §10; these Rev. **1**, 50.] *R. P. Agnew* (Ithaca, N. Y.).

Lyra, Gerhard. Zur Theorie der *C*- und *H*-Summierbarkeit negativer Ordnung. Math. Z. **49**, 538–562 (1944). [MF 11990]

For each complex r not a negative integer, the Cesàro transform $\sigma_n^{(r)}$ of a sequence s_n is

$$(1)\qquad \sigma_n^{(r)}=\sum_{k=0}^{n}\left[\binom{n-k+r-1}{r-1}\Big/\binom{n+r}{r}\right]s_k,$$

and s_n is summable C_r to s if $\sigma_n^{(r)}\to s$. If $r<-1$, some sequences diverging to $+\infty$ are summable C_r, so some authors have modified this definition as follows: if $r<-1$, s_n is summable C_r to s if and only if $\sigma_n^{(r)}\to s$ and $s_n\to s$. If r is a negative integer, the C_r transform in (1) fails to exist. Motivated by the fact that, when r is not a negative integer, the condition $\sigma_n^{(r)}\to s$ may be written in the form

$$\lim_{n\to\infty}\frac{n!}{(n+r)!}\sum_{k=0}^{n}\binom{n-k+r-1}{r-1}s_k=\frac{s}{r!},$$

the author defines s_n to be summable C_r to s when r is a negative integer if

$$\lim_{n\to\infty}\frac{n!}{(n+r)!}\sum_{k=0}^{n}\binom{n-k+r-1}{n-k}s_k=0$$

and $\lim_{n\to\infty}s_n=s$; that is, if $n^{-r}\Delta^{-r}s_n\to 0$ and $s_n\to s$. This definition is equivalent to one given by Hausdorff [Math. Z. 31, 186–196 (1930)]. On the basis of these and related definitions, familiar theorems on Cesàro summability and absolute Cesàro summability are extended to negative integer orders. When r is a negative integer, C_r and C_{-r}^{-1} are equivalent. Theorems on convergence and summability of series of the form $\sum\binom{n+k}{k}\Delta^k a_n$ are given. A sequence s_n is bounded C_{-r} if and only if it is bounded C_r^{-1}. Absolute Cesàro and Hölder methods of the same integer order are equivalent. Finally, theorems on summability of product-sequences A_nB_n and product-series $\sum a_nb_n$ are given.
R. P. Agnew (Ithaca, N. Y.).

Cesco, R. P. On a Tauberian theorem for Nörlund summability. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 180, Vol. 3, num. 4. Serie segunda, 14, Contribuciones, 443–445 (1944). (Spanish) [MF 12193]

Let (N, p_n) be a regular Nörlund method of summability for which $p_n>0$, let $P_n=p_0+\cdots+p_n$, and let, for some constants M and n_0,

$$np_k(P_n-P_{n-k})<MP_nP_k, \quad n>n_0;\ 0\leqq k\leqq n.$$

If $\sum u_n$ is summable (N, p_n) and satisfies the Tauberian condition $u_nP_n/p_n\to 0$, then $\sum u_n$ converges. The proof is similar to a familiar proof of the "small o" C_1 Tauberian theorem, for which $p_n=1$ and $P_n=n+1$. *R. P. Agnew.*

Schur, Issai. On Faber polynomials. Amer. J. Math. 67, 33–41 (1945). [MF 11920]

This posthumous paper deals with the Faber polynomials $P_m(f)$ formally associated with a power series of the form

$$f=f(z)=z\sum_{\nu=0}^{\infty} a_\nu z^{-\nu}, \qquad a_0=1,$$

and defined in the following way. The expansion of $P_m(f)$ in a power series of z has the form

$$P_m(f)=z^m+\sum_{\nu=1}^{\infty} c_{m\nu}z^{-\nu}.$$

The principal problem is to find an explicit representation of $c_{m\nu}$ in terms of the coefficients a_ν of $f(z)$. The author offers a complete solution of this intricate problem based on the following elegant relation between the quantities a_ν and $c_{m\nu}$. Let

$$\left(\sum_{\nu=0}^{\infty} a_\nu z^{-\nu}\right)^m=\sum_{\rho=0}^{\infty} a_{m\rho}z^{-\rho}, \quad m=1, 2, \cdots; a_{m0}=1.$$

We define the matrices A, B, C by $A=(a_{\mu,\mu-\nu})$, $B=(a_{\mu,\mu+\nu})$, $C=(c_{\mu\nu})$, where μ and ν are positive integers, $a_{\mu,0}=1$, $a_{\mu,-k}=0$ $(k\geqq 1)$. Then $B=AC$. From the final result it can be seen that $c_{\mu\nu}$ is a polynomial in the a_ν with nonnegative integer coefficients and $\nu c_{\mu\nu}=\mu c_{\nu\mu}$ [Grunsky's identity, Math. Z. 45, 29–61 (1939)]. *G. Szegö.*

Shtshegloff, M. To the question on the behaviour of a power series on the circle of convergence. Rec. Math. [Mat. Sbornik] N.S. 14(56), 109–132 (1944). (Russian. English summary) [MF 12293]

Let us consider a series $\sum_1^\infty a_n e^{-nt}=f(t)$, convergent for $t>0$, and assume that (i) $a_n\leqq o(1/n)$, (ii) $t_m>t_{m+1}\to +0$, $t_{m+1}/t_m>d>0$. Then $f(t_m)\to S$ implies $f(t)\to S$ as $t\to +0$. The conclusion holds if (i′) $a_n\leqq O(1/n)$, (ii′) $t_m>t_{m+1}\to +0$, $t_{m+1}/t_m\to 1$. The assumption $t_{m+1}/t_m>d>0$ in (ii) cannot be weakened. A less complete result is proved for (ii′). *A. Zygmund* (South Hadley, Mass.).

González, Mario O. On divergent series and analytic extension. Publ. Inst. Mat. Univ. Nac. Litoral 5, 16 pp. (1943). (Spanish) [MF 12093]

Let $\sum a_n z^n$ be a power series with a finite positive radius of convergence. Let $f(z)$ be the function generated by analytic extension along radial lines (half-lines emanating from the origin). A formula is given which sometimes gives values of $f(z)$ on some radial lines in terms of limits of integrals involving values of $f(z)$ on opposite radial lines. The formula is (where the author's $\zeta\to a+i\infty$ may be interpreted to mean $\zeta=a+it$ with $t\to +\infty$ or, alternatively, $t\to -\infty$, but not $|t|\to\infty$)

$$f(z)=\lim_{\zeta\to a+i\infty} \pi^{-1}\sin \pi\zeta \int_0^\infty (1+x)^{-1}x^{\zeta-1}f(-xz)dx,$$

a being a parameter in the interval $0<a<1$. *R. P. Agnew* (Ithaca, N. Y.).

Ríos, Sixto. On the rearrangement of series of functions. I. Dirichlet series with real coefficients. Revista Mat. Hisp.-Amer. (4) 4, 206–209 (1944). (Spanish) [MF 12174]

Let the Dirichlet series $f(s)=\sum a_n e^{-\lambda_n s}$ $(\lambda_n\to\infty)$, with real coefficients, converge (not necessarily absolutely) for $\Re(s)>0$. The author considers a rearranged series in which both the relative order of the positive coefficients and the relative order of the negative coefficients are left undisturbed. Then the rearranged series, if convergent for $\Re(s)>0$, converges to $f(s)$; if it converges uniformly in a larger region, it then provides an analytic continuation of $f(s)$. An example is given. *R. P. Boas, Jr.*

Truesdell, C. Generalizations of Euler's summations of the series $\sum_{n=1}^\infty n^{-2m}$, $\sum_{n=0}^\infty(-)^n(2n+1)^{-2m-1}$, etc. Ann. of Math. (2) 46, 194–195 (1945). [MF 12397]

The author studies the functions defined by the Dirichlet series

$$\frac{(\frac{1}{2}p)^{s-1}}{(p-1)!}\sum_{n=0}^{\infty}\frac{(-1)^{\alpha n}(n+p-1)!}{n!(n+\frac{1}{2}p)^{s+1}},$$

where p and s are positive integers, $\alpha=1$ or 2. Finite expressions are derived for these functions for $\alpha=1$ and s even, and for $\alpha=2$ and $s\equiv p \pmod 2$. The proofs are by induction from classical results of Euler. *I. Niven.*

Thron, W. J. Twin convergence regions for continued fractions $b_0+K(1/b_n)$. Amer. J. Math. 66, 428–438 (1944). [MF 10913]

The author considers the continued fraction (1) $b_0+K[1/b_n]$, where the b_n are complex numbers, and defines regions B_0 and B_1 in the complex plane as twin convergence regions if the conditions $b_{2n}\epsilon B_0$ and $b_{2n+1}\epsilon B_1$ $(n=0, 1, 2, \cdots)$ insure the convergence of (1) provided the necessary conditions for convergence $(\sum|b_n|=\infty$, $b_{2n+1}\neq 0$ for at least one value of $n)$ are also satisfied. The principal theorem is the following. Let the regions B_0 and B_1 be defined by the conditions $re^{i\theta}\epsilon B_0$ if $r\geqq(1+\epsilon)f(\theta)$, $re^{i\theta}\epsilon B_1$ if $r\geqq(1+\epsilon)g(\theta)$, where ϵ is an arbitrarily small positive number and the functions $f(\theta)$ and $g(\theta)$ are positive and finite for $0\leqq\theta\leqq 2\pi$. A necessary condition for B_0 and B_1 to be twin convergence regions is $f(\theta)g(\pi-\theta)\geqq 4$. If $g(\theta)=4/f(\pi-\theta)$, the regions B_0 and B_1 are twin convergence regions if the complements (with respect to the complex plane) of both regions are convex. New convergence theorems for continued fractions (1) emerge, and the author's method provides a new proof for a well-known theorem of Van Vleck [Trans. Amer. Math. Soc. 2, 215–233 (1901)]. *W. Leighton* (Evanston, Ill.).

Thron, W. J. A family of simple convergence regions for continued fractions. Duke Math. J. 11, 779–791 (1944). [MF 11579]

Let A be a set of complex numbers (element set), and $Z(A)$ the set of values of all the approximants of the continued fraction

$$1+\frac{a_1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\cdots$$

for $a_1, a_2, \cdots$ in A (value set). The set A is a convergence set if von Koch's necessary condition for convergence is also sufficient when the a_p are in A. A necessary condition for A to be a convergence set is that $Z(A)$ shall not contain 0 or, what is the same thing, shall not contain two numbers whose sum is unity. With the aid of this criterion, a value region Z depending upon two parameters is chosen, and the corresponding element region A is determined. This is done by making use of the fact that every element in A is of the form $z'(z-1)$, where z' and z are in Z. Actually, regions C for c_p, where $c_p^2=-a_p$, are determined. The author shows that, if an epsilon is introduced in the inequalities defining A, then the resulting smaller region is a convergence region. Moreover, any set which properly contains A is not a convergence set. *H. S. Wall* (Evanston, Ill.).

Lane, Ralph E. The value region problem for continued fractions. Duke Math. J. 12, 207–216 (1945). [MF 12080]

The value region problem for the continued fraction

$$\frac{1}{1+}\frac{a_1}{1+}\frac{a_2}{1+}\cdots \tag{1}$$

is the problem of determining regions E and V in the complex plane such that the approximants of (1) have all their values in V if the a_p have arbitrary values in E; E is called the element region and V the value region of the solution (E, V). The author obtains a general theorem which can be used in solving the value region problem in a comprehensive class of cases. Let $z'=t(u;z)=1/(1+uz)$, and let C, D be arbitrary finite circular regions with centers not at the origin. The theorem in question gives necessary and sufficient conditions on the parameter u in order that $t(u;C)$ shall be contained in D.

By specializing C and D the solutions E: $|z|-\Re(z)\leqq\frac{1}{2}$, V: $|z-1|\leqq 1$ (Scott and Wall); E: $|z|-\Re(z)\leqq t/2$, V: $z+\bar{z}\geqq|z|[2(1-t)^{\frac{1}{2}}+t|z|]$, $0<t\leqq 1$ (Leighton and Thron); E: $|z|\leqq t(1-t)$, V: $|z-1/(1-t^2)|\leqq t/(1-t^2)$, $0<t\leqq\frac{1}{2}$ (Paydon and Wall) and others are readily derived. *H. S. Wall* (Evanston, Ill.).

Lane, Ralph E. The convergence and values of periodic continued fractions. Bull. Amer. Math. Soc. 51, 246–250 (1945). [MF 12261]

The author gives an elementary derivation of the necessary and sufficient conditions of Stolz for convergence of a periodic continued fraction with complex elements,

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_k}{+b_k+}\frac{a_1}{b_1+}\cdots, \qquad a_p\neq 0.$$

This continued fraction may be regarded as generated by iteration of a linear fractional transformation S. If x_1 and x_2 are the fixed points of S, then S^n may be written

$$\frac{1}{z'-x_1}=\frac{1}{z-x_1}+\frac{n}{f-x_1},$$

$f=S(\infty)$, $x_1=x_2$ and finite, or

$$\frac{z'-x_1}{z'-x_2}=\left(\frac{f-x_1}{f-x_2}\right)^n\frac{z-x_1}{z-x_2},$$

$x_1\neq x_2$ and finite. Using these formulas, it is shown that the only cases of convergence are (a) $x_1=x_2$, finite; (b) $F_p\neq x_2$, $p=0, 1, \cdots, k-1$, $|F_{k-1}-x_1|<|F_{k-1}-x_2|$, x_1, x_2 finite, F_p the pth approximant of the continued fraction. When (a) or (b) holds, the value of the continued fraction is x_1. *H. S. Wall* (Evanston, Ill.).

Polynomials, Polynomial Approximations

Rainville, Earl D. A relation between Jacobi and Laguerre polynomials. Bull. Amer. Math. Soc. 51, 266–267 (1945). [MF 12264]

Using generating functions, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is expressed in terms of the Laguerre polynomials of orders α and β:

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}(-1)^k\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(n-k+\alpha+1)}\times L_k^{(\beta)}\left(\frac{1+x}{2}\right)L_{n-k}^{(\alpha)}\left(\frac{1-x}{2}\right).$$

I. M. Sheffer (State College, Pa.).

Rainville, Earl D. Notes on Legendre polynomials. Bull. Amer. Math. Soc. 51, 268–271 (1945). [MF 12265]

The following three identities are derived (the first one is not new):

$$e^{t\cos\theta}J_0(t\sin\theta)=\sum_{n=0}^{\infty}P_n(\cos\theta)t^n/n!;$$

$$P_n(\cos\alpha)=(\sin\alpha/\sin\beta)^n\times\sum_{k=0}^{n}\binom{n}{k}[\sin(\beta-\alpha)/\sin\alpha]^{n-k}P_k(\cos\beta);$$

$$H_n(x)=2^{n+1}x^{n+2}e^{x^2}\int_0^1 e^{-x^2/\beta^2}\beta^{-n-3}P_n(\beta)d\beta$$

(here J_0, P_n, H_n denote, respectively, the Bessel function of order zero, the nth Legendre polynomial, and the nth Hermite polynomial). The proofs make use of generating functions. *I. M. Sheffer* (State College, Pa.).

Rainville, E. D. Certain generating functions and associated polynomials. Amer. Math. Monthly 52, 239–250 (1945). [MF 12442]

The author studies polynomial sets $\{g_n(x)\}$ defined by $e^tG(xt)=\sum_0^\infty g_n(x)t^n/n!$, with $G(x)=\sum_0^\infty a_nx^n/n!$; $\{g_n(x)\}$ has the recurrence property $ng_n(x)=ng_{n-1}(x)+xg_n'(x)$. If $G(x)$ satisfies the linear homogeneous differential equation

$$\sum_{k=0}^{m}x^kF_k(\theta)G(x)=0, \qquad \theta\equiv x(d/dx),$$

where $F_0, \cdots, F_m$ are polynomial operators in θ, then the set $\{g_n(x)\}$ satisfies the differential equation

$$\sum_{k=0}^{m}(-1)^kx^kF_k(\theta)(\theta-n)_kg_n(x)=0, \qquad n\geqq m.$$

Here $(c)_k\equiv c(c+1)\cdots(c+k-1)$, $(c)_0=1$.

Particular cases are examined, among them the Laguerre polynomials and hypergeometric polynomials, the latter corresponding to

$$G(x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{x^n}{n!}$$

(the Pochhammer-Barnes confluent hypergeometric function). *I. M. Sheffer* (State College, Pa.).

Charadze, A. K. On a generalization of Jacobi's polynomials. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 495–502 (1943). (Georgian and Russian) [MF 11705]

Generalizing the differential equation of Jacobi polynomials orthogonal in $(-1, 1)$, and defining orthogonality on a symmetrical pencil P_k consisting of k rectilinear segments joining the origin to the points $1, \omega, \omega^2, \cdots, \omega^{k-1}$ $(\omega=e^{2\pi i/k})$ with the aid of the linear operator I_k:

$$I_k[F'(z)]=\sum_{m=0}^{k-1} F(\omega^m)\cdot\omega^{k-m},$$

the author introduces and studies generalized Jacobi polynomials, orthogonal on P_k, of degrees mk and $mk+1$ for $m=0, 1, 2, \cdots$. If $k=3$ they are related to elliptic functions in the same way as Chebyshev's polynomials $\cos(n \arccos z)$ are related to the trigonometric functions in the case $k=2$.

E. G. Kogbetliantz (New York, N. Y.).

Thorne, C. J. A property of Appell sets. Amer. Math. Monthly 52, 191–193 (1945). [MF 12256]

It is shown that a set of polynomials $\{\phi_n(x)\}$, $n=0, 1, \cdots$, is an Appell set $(\phi_n'(x)=\phi_{n-1}(x))$ if and only if it has the property

$$\int_0^\infty \phi_n^{(r)}(x)d\alpha(x)=\begin{cases}1, & n=r,\\ 0, & n\neq r,\end{cases}$$

where $\alpha(x)$ is a function, of bounded variation on $(0, \infty)$, whose moment constants $\mu_n=\int_0^\infty x^n d\alpha(x)$ all exist, with $\mu_0\neq 0$.

I. M. Sheffer (State College, Pa.).

Schmidt, Erhard. Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum. Math. Ann. 119, 165–204 (1944). [MF 11894]

Define the three quantities M_n, M_n^*, M_n^{**} by

$$M_n^2=\max\left\{\int_{-1}^1[F'(x)]^2dx\Big/\int_{-1}^1[F(x)]^2dx\right\},$$

$$M_n^{*2}=\max\left\{\int_0^\infty e^{-x}[F'(x)]^2dx\Big/\int_0^\infty e^{-x}[F(x)]^2dx\right\},$$

$$M_n^{**2}=\max\left\{\int_{-\infty}^\infty e^{-x^2}[F'(x)]^2dx\Big/\int_{-\infty}^\infty e^{-x^2}[F(x)]^2dx\right\},$$

where $F(t)\not\equiv 0$ runs through all real polynomials of degree not exceeding n. It is shown that in each case the maximum is attained for a polynomial which is unique to within an arbitrary constant factor, and the following asymptotic expressions are obtained: $M_n^{**}=(2n)^{\frac{1}{2}}$;

$$M_n=\frac{(n+\frac{3}{2})^2}{\pi}\left\{1-\frac{\pi^2-3}{12(n+\frac{3}{2})^2}+\frac{R_n}{(n+\frac{3}{2})^4}\right\}^{-1};$$

$$M_n^*=\frac{2n+1}{\pi}\left\{1-\frac{\pi^2}{24(2n+1)^2}+\frac{R_n}{(2n+1)^4}\right\}^{-1};$$

in the expression for M_n, $n\geqq 5$ and $-6<R_n<13$; for M_n^*, $n\geqq 2$ and $-8/3<R_n<4/3$.

The method of proof makes use of properties of the Legendre, Laguerre and Hermite polynomials. For M_n (to give only one case), the maximizing value is given by $M_n=1/\lambda_{k,1}$ or $1/\sigma_{k,1}$ according as $n=2k$ or $n=2k-1$, where $\lambda_{k,1}$, $\sigma_{k,1}$ are defined as follows. Let $\{p_n(t)\}$ be the orthonormal Legendre polynomials, and set

$$\alpha_{rp}=\int_{-1}^1 p_{2r}'(t)p_{2p}'(t)dt,\quad \beta_{rp}=\int_{-1}^1 p_{2r-1}'(t)p_{2p-1}'(t)dt.$$

Then $\sum_1^k\alpha_{rp}x_rx_p$, $\sum_1^k\beta_{rp}x_rx_p$ are positive definite forms, and these can be reduced, each by a suitable orthogonal transformation, to $\sum_1^k y_r^2/\lambda_{k,r}^2$, $\sum_1^k y_r^2/\sigma_{k,r}^2$, where the λ's and σ's (the numbers of each set being positive and distinct) are taken in ascending order of magnitude. These same transformations carry the polynomials $p_2(t), \cdots, p_{2k}(t)$ into polynomials $g_{k,1}(t), \cdots, g_{k,k}(t)$, and $p_1(t), \cdots, p_{2k-1}(t)$ into polynomials $h_{k,1}(t), \cdots, h_{k,k}(t)$. The g's are all even and of degree $2k$, and the h's are all odd and of degree $2k-1$. Moreover, the maximizing polynomial of degree (not exceeding and, in fact, actually equal to) n is $Cg_{k,1}(t)$ if $n=2k$ and $Ch_{k,1}(t)$ if $n=2k-1$, C being an arbitrary constant.

I. M. Sheffer (State College, Pa.).

Peiser, Alfred M. Some applications of Fourier analysis and calculus of probability to the study of real roots of algebraic equations. Trans. Amer. Math. Soc. 56, 470–493 (1944). [MF 11491]

Let $F=F(x, t)=x_0+x_1t+\cdots+x_nt^n=0$ be an algebraic equation with real coefficients. Denote by

$$N_n(x)=N_n(x_0, x_1, \cdots, x_n)$$

the number of its real roots. The author proves among others the following result. Let $K_n=\sum_{j=0}^n t^{2j}$, $L_n=\sum_{j=0}^n jt^{2j-1}$, $M_n=\sum_{j=0}^n j^2t^{2j-2}$, $D_n=K_nM_n-L_n^2$, $R_n=D_n^{\frac{1}{2}}/K_n$. Furthermore, let

$$X_n=\sum_{j=0}^n t^jx_j/K_n^{\frac{1}{2}},\quad Y_n=\sum_{j=0}^n (K_njt^{j-1}-L_nt^j)x_j/(K_nD_n)^{\frac{1}{2}}.$$

Then, at each continuity point of N_n,

$$N_n(x)=\lim_{h\to\infty} h/(2\pi)^{\frac{1}{2}}\int_{-\infty}^\infty R_n|Y_n|\exp\left[-x_n^2h^2/2\right]dt.$$

P. Erdös (Ann Arbor, Mich.).

Special Functions

Häusermann, A. Über die Berechnung singulärer Moduln bei Ludwig Schläfli. Vierteljschr. Naturforsch. Ges. Zürich 89, 216–218 (1944). [MF 12241]

Short reviews of eight Swiss papers (1911–1943) on the subject.

Bancroft, T. A. Note on an identity in the incomplete beta function. Ann. Math. Statistics 16, 98–99 (1945). [MF 12363]

The following identity involving the incomplete beta function is proved:

$$(p+q+n-1)^{(n)}I_x(p, q)=\sum_{r=0}^n \binom{n}{r}(p+n-r-1)^{(n-r)}\times(q+r-1)^{(r)}I_x(p+n-r, q+r),$$

where $I_x(p, q)$ stands for $B_x(p, q)/B(p, q)$ and

$$y^{(n)}=y(y-1)(y-2)\cdots(y-n+1),$$

in accordance with the customary factorial notation. [The

author uses, without explicit definition, the less usual notation $y^{[n]}$, which has also been employed by some writers to denote a central factorial.] *T. N. E. Greville.*

Copson, E. T. An integral formula for $Q_n(\cos\theta)$. Proc. Edinburgh Math. Soc. (2) 7, 81–82 (1945). [MF 12350]

The following representation of Legendre's function of the second kind is obtained:

$$Q_n(\cos\theta)=\tfrac{1}{4}i^{n+1}\int_{-\pi}^{\pi}|\sin^n u|(\sin\theta+i\cos\theta\sin u)^{-n-1}du,$$

$0<\theta<\pi$.

F. G. Dressel (Durham, N. C).

Erdélyi, A., and Kermack, W. O. Note on the equation $f(z)K_n'(z)-g(z)K_n(z)=0$. Proc. Cambridge Philos. Soc. 41, 74–75 (1945). [MF 12380]

The authors prove that the equation of the title has no roots in the right half of the complex z-plane provided that $\Re[g(z)/f(z)]\geqq 0$ when $\Re(z)\geqq 0$. *M. C. Gray.*

Rice, S. O. Sums of series of the form $\sum_0^\infty a_nJ_{n+\alpha}(z)J_{n+\beta}(z)$. Philos. Mag. (7) 35, 686–693 (1944). [MF 11958]

The series $\sum_0^\infty a_nJ_{\alpha\pm n}(z)J_{\beta+n}(z)$ are transformed into double series by substitution of the known power series expansions [cf. G. N. Watson, Theory of Bessel Functions, Cambridge University Press, Cambridge, England, 1922, p. 147] of $J_{\alpha\pm n}(z)J_{\beta+n}(z)$. Then a_n is chosen so that one of the two summations can be carried out. One of the choices is $\alpha=\frac{1}{2}$, $\beta=\pm\frac{1}{2}$, $a_{2m}=0$, $a_{2m+1}=(4m+3)/(m+1)(2m+1)$; it leads to a proof of two relations given without rigorous direct proof by S. A. Schelkunoff [Proc. I.R.E. 29, 493–521 (1941)]. *A. Erdélyi* (Edinburgh).

Bell, R. P. Eigen-values and eigen-functions for the operator $d^2/dx^2-|x|$. Philos. Mag. (7) 35, 582–588 (1944). [MF 11799]

The equation $\psi''-|x|\psi=-\lambda\psi$ has a known solution in terms of Bessel functions of order $\pm\frac{1}{3}$ if the solution is required to satisfy appropriate boundary conditions; the author shows that the eigenvalues λ_n are of the form $(\frac{3}{2}\mu)^{\frac{2}{3}}$, where μ is the root of $J_{2/3}(\mu)-J_{-2/3}(\mu)=0$ or $J_{1/3}(\mu)+J_{-1/3}(\mu)=0$ according as n is even or odd. The odd eigenvalues are obtainable from existing tables; the even ones, except for λ_0, are obtainable from asymptotic formulas; λ_0 is obtained from a 4-place table which the author gives for $J_{2/3}(z)$ and $J_{-2/3}(z)$ from $z=0$ to $z=1$ at intervals of .02. The equation $\psi''-|x|^q\psi=-\lambda\psi$ determines the wave-functions and energy levels for an oscillator with energy $V(y)=a|y|^q$; this equation has been solved for $q=1$ [present paper], $q=2$ [classical] and $q=4$ [see the following review]. The author suggests that these results could be usefully applied to obtain approximate energy levels for other values of q by interpolation.

R. P. Boas, Jr. (Providence, R. I.).

Bell, R. P. The occurrence and properties of molecular vibrations with $V(x)=ax^4$. Proc. Roy. Soc. London. Ser. A. 183, 328–337 (1945). [MF 12051]

The differential equation

$$\psi''(\xi)+(\lambda-\xi^4)\psi(\xi)=0$$

is (after changes of variable) the wave equation for an oscillator with potential energy $V(x)=ax^4$. It is integrated numerically; the first four eigenvalues are computed and the eigenfunctions are exhibited graphically. The author discusses physical problems which lead to his equation, and physical consequences of his results. *R. P. Boas, Jr.*

Shanker, Hari. Some infinite integrals involving cylinder functions. J. Indian Math. Soc. (N.S.) 8, 57–60 (1944). [MF 12247]

The Hankel transforms of order ν of the functions $x^{\nu\pm 1}D_n(ix)D_{-n-2m-1}(x)$ are obtained in the form of finite series of confluent hypergeometric functions when n is a positive integer and the other parameters are suitably restricted. Other integrals involving products of parabolic cylinder functions and generalized Laguerre polynomials are also evaluated. *M. C. Gray* (New York, N. Y.).

MacRobert, T. M. Some formulae for the E-function. Philos. Mag. (7) 31, 254–260 (1941). [MF 11752]

The E-function is an extension of the hypergeometric series. If $p<q+1$ and $x\neq 0$, or $p=q+1$ and $|x|>1$,

$$E(p;\alpha_r:q;\rho_s:x)$$
$$=(\prod\Gamma(\alpha)/\prod\Gamma(\rho))\,{}_pF_q(\alpha_1,\cdots,\alpha_p;\rho_1,\cdots,\rho_q;-1/x);$$

if $p>q+1$ the right hand side gives the asymptotic expansion of the E-function. The author proves

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha_{p+1}-1}E(p;\alpha_r:q;\rho_s:x/\lambda)d\lambda=E(p+1;\alpha_r:q;\rho_s:x)$$

and three other formulae for "integrals of the second kind," the formula

$$\int_0^\infty \lambda^{\rho_{q+1}-\alpha_{p+1}-1}(1+\lambda)^{-\rho_{q+1}}E(p;\alpha_r:q;\rho_s:(1+\lambda)x)d\lambda$$
$$=\Gamma(\rho_{q+1}-\alpha_{p+1})E(p+1;\alpha_r:q+1;\rho_s:x)$$

for an "integral of the first kind" (the terminology refers to the analogous Euler integrals), and various recurrence relations.

Special cases of the formulae derived in this paper are shown to lead to interesting relations between Bessel functions, Legendre functions and confluent hypergeometric functions. *A. Erdélyi* (Edinburgh).

Sinha, S. Infinite integrals involving Bessel functions of imaginary argument. J. Indian Math. Soc. (N.S.) 8, 21–26 (1944). [MF 11274]

The object of this paper is to evaluate the infinite integral

$$\int_0^\infty x^{s-1}K_m(ax)\,{}_pF_q(-b^2x^2)dx$$

in terms of ${}_{p+2}F_q(-4b^2/a^2)$. [The parameters of the generalized hypergeometric series ${}_pF_q$ and ${}_{p+2}F_q$ have been omitted.] The evaluation is performed by means of two lemmas. [In the statement of these lemmas there are mistakes. Lemma 1 should read: "When $p-q\leqq -1$, $I_1=\cdots$," and lemma 2 should read: "When $R(s\pm m+\frac{1}{2})>0$ and either $p-q\leqq -1$ or $p=q$ and $c>-1$, then $I_2=\cdots$."] The result of this evaluation is in fact known, in a more general form [cf. the preceding review]. A considerable number of integrals, not all new, are derived as particular cases of the main result.

The integrands contain Bessel, Struve and confluent hypergeometric functions. *A. Erdélyi* (Edinburgh).

Eriksson, H. Adolf S. Some applications and properties of the hyperspherical harmonics with three polar angles. Ark. Mat. Astr. Fys. **30B**, no. 4, 8 pp. (1944). [MF 12007]

Let P, P' be two points on the four-dimensional unit hypersphere with center O, and let T be the angle POP'. In section 1 of this paper the author expands cosec $\frac{1}{2}T$ in a series of hyperspherical surface harmonics $\psi_{n,l,m}(\alpha, \theta, \phi)$, where α, θ, ϕ are hyperspherical polar coordinates on the surface of the unit hypersphere.

If Q, Q' are two points in ordinary three-dimensional space, (r, θ, ϕ) and (r', θ', ϕ') the spherical polar coordinates of these points, $r=\cot \frac{1}{2}\alpha$, $r'=\cot \frac{1}{2}\alpha'$, and P and P' the points (α, θ, ϕ), $(\alpha', \theta', \phi')$ on the four-dimensional unit hypersphere, the reciprocal (three-dimensional) distance of Q and Q' is equal to $\sin \frac{1}{2}\alpha \sin \frac{1}{2}\alpha'$ cosec $\frac{1}{2}T$ and hence can be expanded in a series of hyperspherical harmonics.

In section 2, the author infers from the expansion of cosec $\frac{1}{2}T$ that the hyperspherical surface harmonics $\psi(P)$ satisfy the integral equation

$$\psi_{nlm}(P) = (n^2 - \tfrac{1}{4})/(8\pi) \int \operatorname{cosec} \tfrac{1}{2}T \psi_{nlm}(P')d\omega',$$

the integral being extended over the surface of the unit hypersphere. [Actually, hyperspherical surface harmonics satisfy integral equations the nucleus of which is any function of T; cf. Math. Ann. **115**, 456–465 (1938).]

In section 3 a brief discussion is given of the Kepler problem for the hydrogen atom in momentum space. The Schrödinger equation

$$(p^2 - 2mE)\psi = 2r^{-1}me^2\psi$$

is left-multiplied by r, iterated (that is, applied twice in succession) and then transformed into momentum space. The resulting differential equation can be converted into an integral equation whose characteristic functions are expressed in terms of hyperspherical harmonics, thus obtaining agreement with Fock's well-known result.

A. Erdélyi (Edinburgh).

Kotowski, Gertrud. Lösungen der inhomogenen Mathieuschen Differentialgleichung mit periodischer Störfunktion beliebiger Frequenz (mit besonderer Berücksichtigung der Resonanzlösungen). Z. Angew. Math. Mech. **23**, 213–229 (1943). [MF 11743]

In the first part of this paper the solutions of

$$(1) \qquad d^2y/dt^2 + (\omega^2 + a^2 \cos \Omega t)y = f(p, t)$$

are discussed, where f is a periodic function of t with period $T = 2\pi/p$. Detailed results are given for the case $f = e^{ipt}$. By the method of variation of parameters it is shown that a particular solution of (1) (the forced vibration) is of the form e^{ipt} multiplied by a periodic function (of period $2\pi/\Omega$) of t provided that $p/\Omega \pm i\mu$ is not an integer (μ being the characteristic exponent associated with the Mathieu differential equation $d^2y/dx^2 + \{(\omega/\Omega)^2 + (a/\Omega)^2 \cos x\}y = 0$).

In the case of resonance ($p/\Omega \pm i\mu$ an integer) the particular integral increases indefinitely. The Fourier series for the particular integral are given for the stable region ($2i\mu$ real and not an integer), the unstable region ($2i\mu$ complex with integral real part), and for the boundary curves between these two regions ($2i\mu$ an integer). The solution for a more general periodic function $f(p, t)$ can be obtained by superposition. There is a numerical discussion of the first three resonance intervals for $f = a_0 + a_1 \cos pt + a_2 \cos 2pt$.

In the second part of the paper the influence of attenuation is discussed: the left hand side of (1) is amplified by addition of the term $2\Omega\delta dy/dt$. The condition for resonance is now that $p/\Omega + i(\delta \pm \mu)$ is an integer; this condition can never be satisfied in the stable region (where $i\mu$ is real). In the unstable region (where the imaginary part of μ is half an integer), resonance may occur when p/Ω is half an integer and the real part of μ is equal to δ. As in the former case, the particular solution is bounded except in the case of resonance. There is also a numerical discussion of the first three unstable intervals.

The theoretical discussion in this paper is based on, and in fact to some extent repeats parts of, a paper by the reviewer [Arch. Elektrotechnik **29**, 473–489 (1935)]. The author points out a mistake in that paper: the function denoted by $g(pt; \Omega)$ in the top line of page 487 vanishes identically and the particular solution remains finite for all values of t. *A. Erdélyi* (Edinburgh).

Zeilon, Nils. On the theory of the Stark effect. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 11, 20 pp. (1944). [MF 11680]

The Schrödinger equation for the Stark effect in hydrogen, written in parabolic coordinates,

$$(1) \quad U'' + \{-\tfrac{1}{4} + (n + \tfrac{1}{2}(m+1))x^{-1} - \tfrac{1}{4}(m^2-1)x^{-2}\}U = \pm\lambda x U,$$

presents difficulties due to the presence of the right hand side. Oseen [Ark. Mat. Astr. Fys. **25A**, no. 2 (1934)] has proposed replacing this right hand side by $\pm\lambda x(1+\sigma x)^{-2}U$, which vanishes at infinity and is practically identical with $\pm\lambda x U$ near the origin. Such a change facilitates the exact definition of characteristic values and characteristic functions.

The author now replaces $\pm\lambda x U$ by

$$\pm\lambda x U/(1+\beta_1 x + \beta_2 x^2)$$

on the right hand side of (1) and shows that β_1 and β_2 can be so determined that the modified equation has a solution of the form of a polynomial in x multiplied by

$$e^{-\frac{1}{2}x}x^{\frac{1}{2}(m+1)}(1+\beta_1 x + \beta_2 x^2).$$

For the determination of β_1, β_2 and the coefficients of the polynomial a nonlinear system of algebraic equations is obtained and the author gives an approximate solution of this system under the assumption that λ, β_1, β_2 and the deviation of n from an integer are small. In this way he reproduces the formula for the change in the quantum number in the theory of the Stark effect of the first order.

The method is generalized for the case in which the right hand side of (1) is replaced by

$$\pm\lambda x \frac{\beta_0 + \beta_1 x + \cdots + \beta_{2p-2}x^{2p-2}}{\beta_0 + \beta_1 x + \cdots + \beta_{2p}x^{2p}} U.$$

An investigation is given of the ensuing (nonlinear) system of algebraic equations for the coefficients of the polynomial appearing in the solution of (1) with this right hand side. Numerical results are also given. *A. Erdélyi*.

GEOMETRY

Hadamard, J. On the three-cusped hypocycloid. Math. Gaz. 29, 66–67 (1945). [MF 12537]

Rolfe, Kathryn B. A geometrical interpretation of the invariant system of two binary cubics. Nat. Math. Mag. 19, 211–220 (1945). [MF 12437]

Chariar, V. R. On scrolls generated by lines whose polars with regard to a pencil and a net of quadrics are concurrent. Bull. Calcutta Math. Soc. 36, 122–124 (1944). [MF 12409]

Rao, C. V. H. On a problem in solid geometry. Bull. Calcutta Math. Soc. 36, 132–134 (1944). [MF 12411]

The problem under reference is to determine the lengths of the axes of an arbitrary plane section of the general quadric surface. *Extract from the paper.*

Segre, B. A four-dimensional analogue of Pascal's theorem for conics. Amer. Math. Monthly 52, 119–131 (1945). [MF 12063]

Two sets $a_1, \cdots, a_4$ and $b_1, \cdots, b_4$ of four skew lines are said to form a double-four if a_i meets b_j whenever $i\neq j$, but a_i and b_i are skew. Part I of this paper proves that a double-four in ordinary space (complex projective 3-space) lies on a cubic surface if and only if the lines contain respective points $A_1, \cdots, A_4$ and $B_1, \cdots, B_4$ such that the four points A_i, A_j, B_h, B_k are coplanar for every permutation $ijhk$ of 1234. Five lines in 4-space do not generally lie on a quadric 3-fold; if they do, they are said to be related. Part II provides linear criteria for related quintuplets of lines. One criterion is as follows. Five lines $a_1, \cdots, a_5$ are related if and only if the pairs of hyperplanes a_1a_2 and a_3a_4, a_1a_3 and a_4a_2, a_1a_4 and a_2a_3 cut a_5 in a quadrangular set of points. Part III deals with the analogous concept of related quintuplets of circles (in Euclidean 3-space) and ends with a property of Kasner's "turbines." *H. S. M. Coxeter.*

***Valeiras, Antonio. Triangle of minimum perimeter inscribed in another in non-Euclidean geometries.** Memorias sobre Matematicas (1942–44) por Antonio Valeiras, pp. 11–23. Buenos Aires, 1944. (Spanish) [MF 12381]

The paper originally appeared in Publ. Circulo Mat. Inst. Nac. Profesorado Secund. no. 6 (1942); cf. these Rev. 5, 9, where the title was incorrectly translated.

Scherk, Peter. On differentiable arcs and curves. IVa. On certain singularities of curves of order $n+1$ in projective n-space. Ann. of Math. (2) 46, 175–181 (1945). [MF 12395]

[Part IV appeared in the same Ann. (2) 46, 68–82 (1945); these Rev. 6, 183.] Let K be a closed curve of order $n+1$ in n-dimensional projective space. In part IV the author defined multiplicities under certain differentiability conditions. Let N_ν^n denote the number of special ν-dimensional linear spaces (L_ν), that is, those L_ν which have $\nu+2$ intersections with K but no sub-$L_{\nu-1}$ of which has $\nu+1$ intersections with K. If M_ν^n denotes the number of osculating L_ν which intersect K at a second point, it is shown that

$$\sum_{\nu=0}^{n-1}(n-\nu)N_\nu^n+\sum_{\nu=0}^{n-2}(n-\nu-1)M_\nu^n<\tfrac{1}{4}n^2+\tfrac{3}{4}n.$$

H. Busemann (Chicago, Ill.).

Robinson, H. A. A problem of regions. Amer. Math. Monthly 52, 33–34 (1945). [MF 11909]

A formula $S(n, d, k)=\sum_{i=0}^{d}\binom{n}{i}+(k-1)\binom{n}{d}$ is given for the number of d-cells formed by the partitioning of a simply connected Euclidean region R_d by n "hyperplanes" of $d-1$ dimensions, each intersecting each other in k $(d-2)$-cells within R_d, no $d+1$ hyperplanes passing through a $(d-2)$-cell; there is another formula $C(n, d, k)=\binom{n-1}{d}+(k-1)\binom{n}{d}$ for the corresponding number of interior regions formed. The author apparently uses the term "hyperplane" to refer to nonlinear curves or hypersurfaces when $k>1$. The special case $S(n, d, k)=2^n$ applies for $n<d$, not for $n>d$ as printed.

The formulas appear to be correct for $k=1$ but need further clarification in at least certain instances when $k\geqq 2$. For instance, consider the plane partitioned by the three parabolas $y=x^2$, $y=2x^2-1$ and $y=3x^2-3$. We have $S(2, 2, 2)=10$. Now if these same equations are thought of as representing intersecting cylindrical surfaces in space we should have $S(3, 3, 2)=10$, which does not agree with the value 9 given by the author's formulas. Possibly the author intends that the complete intersections of his "hyperplanes" should lie within a bounded region, which would exclude the cylindrical region of the example cited. *J. S. Frame* (East Lansing, Mich.).

Differential Geometry

***Valeiras, Antonio. An elementary deduction of the equations of the loxodrome and some new properties of the stereographic projection.** Memorias sobre Matematicas (1942–44) por Antonio Valeiras, pp. 51–58. Buenos Aires, 1944. (Spanish) [MF 12385]

If a sphere is projected stereographically on a plane, the great circles through the pole map into a pencil of straight lines; a logarithmic spiral cuts all these lines at the same angle; hence the equations of a loxodrome on the sphere are immediately obtained from the equation of the spiral. The author observes, furthermore, that the equations of a stereographic projection take a very compact form in terms of the parameters $u=\log\rho$, $v=\theta$ if (ρ, θ) are polar coordinates in the plane; he uses this form to discuss properties of the projection. *R. P. Boas, Jr.* (Providence, R. I.).

Kasner, Edward, and DeCicco, John. Irregular projective invariants. Proc. Nat. Acad. Sci. U.S.A. 31, 123–125 (1945). [MF 12322]

The authors show that an irregular analytic element with a simple cusp possesses two relative invariants of the seventh and eighth orders and therefore an absolute invariant of the eighth order with respect to the group of plane collineations. *A. Fialkow* (New York, N. Y.).

De Cicco, John. The magnilong near-Laguerre transformations. Nat. Math. Mag. 19, 229–235 (1945). [MF 12439]

Bachvaloff, S. Sur un invariant des transformations asymptotiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 87–88 (1944). [MF 12027]

Let L_1 and L_2 be two stratifiable congruences generated by lines l_1 and l_2; θ is the angle between corresponding lines,

$2a$ is the length of their common perpendicular, K is the Gaussian curvature of a stratifying surface M_i and α_i is the angle which the tangent plane to M_i makes with the above common perpendicular. It is shown that

$$(2a/\sin\theta)^4 = ((\cos^4\alpha_0)/K_0)((\cos^4\alpha_1)/K_1),$$

where M_0, M_2 belong to L_1 and M_1, M_3 to L_2. From this it follows that each factor of the last equation is an invariant of the corresponding congruence, that is,

$$K_0 = (\cos^4\alpha_0)/I_0.$$

M. S. Knebelman (Pullman, Wash.).

Wilkins, J. Ernest, Jr. The contact of a cubic surface with a ruled surface. Amer. J. Math. 67, 71–82 (1945). [MF 11923]

The author investigates contacts of higher order than four of families of cubic surfaces f_i with a ruled surface S at a generic point of S. He finds that, if S is developable, there is a three-parameter family of f_i with contact of order five, a unique f with contact of order six and no f with contact of order seven. If S is nondevelopable, the f_i with contact of order five form a one-parameter family and there is no f with contact of order six, other than S itself when S is a cubic. *T. R. Hollcroft* (Aurora, N. Y.).

Robinson, Charles V. A simple way of computing the Gauss curvature of a surface. Rep. Math. Colloquium (2) 5–6, 16–24 (1944). [MF 11402]

There is a rational function of the six distances between pairs of points in a semi-dependent quadruple which may be used to replace the embedding curvature of the quadruple in defining the Gauss curvature of a surface at a point. *J. L. Dorroh* (Baton Rouge, La.).

Botella Raduán, F. On the analytic expression of the curvature of a Riemannian space. Revista Mat. Hisp.-Amer. (4) 3, 302–309 (1943). (Spanish) [MF 12156]

The author examines the conditions for absolute parallelism in a Riemann space; this means that parallel displacement of any vector from P_0 to P_1 is independent of the path joining P_0 to P_1. The author disagrees with the standard result that absolute parallelism requires the vanishing of the Riemann-Christoffel tensor. His proposed substitute for this condition is stated in noninvariant form and is not substantiated by his proof. *C. B. Allendoerfer.*

Lichnerowicz, André. Sur une inégalité relative aux espaces riemanniens complètement harmoniques. C. R. Acad. Sci. Paris 218, 436–437 (1944). [MF 12114]

According to Copson and Ruse [Proc. Roy. Soc. Edinburgh 60, 117–133 (1940); these Rev. 2, 20] a Riemannian space R_n is called completely harmonic if for each point P of R_n the geodesic distance s from P has the property that $\Delta_2(\frac{1}{2}s^2) = f(\frac{1}{2}s^2)$, Δ_2 being the second differential parameter. It is not known whether every completely harmonic Riemannian space is of constant curvature. The author proves that it is of constant curvature if $f''(0) \geqq -20(n-1)f''(0)$, and that then only the equality sign in this condition is is possible. *S. Chern* (Princeton, N. J.).

Lichnerowicz, André. Sur les espaces riemanniens complètement harmoniques. C. R. Acad. Sci. Paris 218, 493–495 (1944). [MF 12118]

Concerning completely harmonic Riemannian spaces H_n [cf. the preceding review] the following two results are proved. (1) If H_n is not Euclidean, it is not the product space of two Riemannian spaces. (2) If H_n is a hypersurface of a space of constant curvature, it is itself of constant curvature. *S. Chern* (Princeton, N. J.).

Chern, Shiing-Shen. On a theorem of algebra and its geometrical application. J. Indian Math. Soc. (N.S.) 8, 29–36 (1944). [MF 12244]

The algebraic theorem states that a set of linearly independent quadratic forms is determined, up to an orthogonal transformation, by certain sums of exterior products of the first partial derivatives of the forms, if enough of those derivatives are independent. The geometric application is a proof (with the method of moving frames) of Allendoerfer's theorem that two m-dimensional varieties in n-space are congruent if they are isometric, their first normal spaces have the same dimensions; and one of them is of type at least 3. The algebraic theorem is applied to the quadratic forms which make up the second fundamental forms of the varieties; the exterior products of the partial derivatives appear in connection with the "equations of structure" of the group of motions. *H. Samelson.*

Choudhury, A. C. On 2-webs of curves in R_n. J. Indian Math. Soc. (N.S.) 8, 36–44 (1944). [MF 12245]

With a two-web of curves in a three-dimensional space, which does not span a family of surfaces, there is associated an ordinary differential equation of the second order. The author generalizes this theorem to a two-web in an n-dimensional space, obtaining an ordinary differential equation of order $n-1$. *S. Chern* (Princeton, N. J.).

Grove, V. G. Quadrics associated with a curve on a surface. Bull. Amer. Math. Soc. 51, 281–287 (1945). [MF 12269]

The author discusses a three-parameter set of quadrics associated with a point of a curve C on a surface S in projective 3-space. He finds it convenient to split it into one-parameter families $Q(l_2, m_2, l_3, m_3)$ with parameter k, each family determined by four constants. A remarkable property of these quadrics is that they include many familiar quadrics associated with a curve (those of Darboux, Moutard, Wu, and others). The families are given an invariant characterization among the quadrics having second order contact with S and it is shown that the most general transformation of Čech can be interpreted as polarization with respect to them. Two values of the parameter k are selected invariantively and in a geometric fashion to give two quadrics Q_u, Q_v of special interest. For example, $Q_u(-1, 0, 1, 0)$, $Q_v(0, 1, 0, -1)$ are the asymptotic osculating quadrics of C, and $Q_u(0, h, h, 0)$, $Q_v(0, h, h, 0)$ are conjugal quadrics. The author is led to define $Q_u(0, 1, 0, -1)$, $Q_v(-1, 0, 1, 0)$ as anti-asymptotic osculating quadrics of C. They intersect in the asymptotic tangents and in a conic whose plane passes through the projective normal when and only when C is pangeodesic. A number of other applications of the new quadric family are given, much use being made of the notion of R_k-associate of a line due to P. O. Bell. *J. L. Vanderslice* (College Park, Md.).

Buckel, Walter. Über eine Verallgemeinerung der Dupinschen Indikatrix. J. Reine Angew. Math. 185, 144–191 (1943). [MF 12107]

This paper begins with a study of the intersection and contact properties of an n-dimensional manifold of class C^{n+1} in Euclidean $(n+1)$-space with lines in that space, attention being restricted to the neighborhood of a point. The first results concern the relation between the degree of contact of a line tangent to the manifold and the number of intersections of nearby lines with the manifold. By means of intersection properties there are associated with a point of the manifold an order and an "ordergeometric indicatrix" of length L, finite or infinite. In general the length L and the order are equal. An affine generalized Dupin indicatrix, which is a system of equations, is defined by means of the Taylor expansion of the distance from the manifold to a line tangent to the manifold and its relationship to the order is determined.

To a limited extent, the results generalize to subsets of a topological space S, where again the intersection properties are the determining considerations. In the general case, the line is replaced by the image of a given set K. If the set S is E_n and the set of transformations defining the images are the rigid motions of E_n, the indicatrix obtained is called the kinematic indicatrix. This concept is applied specifically to the case where the set K is a differentiable arc, and thus intersection and contact properties of curves and manifolds are obtained. *G. A. Hedlund* (Charlottesville, Va.).

v. Sz. Nagy, Gyula (Julius). Kurven von Maximalindex in mehrdimensionalen projektiven Räumen. J. Reine Angew. Math. 186, 30–39 (1944). [MF 12099]

A curve of order n (the order is the maximum number of points of intersection with a hyperplane) in real projective q-space is said to be of one cycle (einzügig) if it is the continuous image of a circle, and of s cycles if it is composed of s curves of one cycle. The index i (the minimum number of points of intersection with a hyperplane) of a differentiable curve is shown to satisfy the fundamental relation $i \leqq n-q$ for q even, $i \leqq n-q+1$ for q odd. Those curves $K_n{}^q$, of one or many cycles, for which the equality sign holds, are said to have maximal index. The author proves that, in general, a curve $K_n{}^q$, on being projected from a tangent onto a $(q-2)$-space, becomes a curve K_{n-2}^{q-2}. He also gives conditions under which a projection from a point is again a curve of maximal index. By use of these projections the results of a series of earlier papers for curves of maximal index in 2- and 3-space are extended to q-space. One of these results is that a curve $K_n{}^q$ is at most of $n+1-q$ cycles. The paper is concluded by the construction of an algebraic curve $K_n{}^q$ of exactly $n+1-q$ cycles. *D. Derry.*

v. Sz. Nagy, Gyula (Julius). Algebraische Kurven vom Maximalindex im mehrdimensionalen Raum. J. Reine Angew. Math. 186, 40–48 (1944). [MF 12100]

This paper is concerned with two existence proofs. Let $i_1, i_2, \cdots, i_s$ be integers which, for odd q, are positive with sum $n-q+1$ and, for even q, are positive with one possible exception and with sum $n-q$. Then an algebraic curve of order n in projective q-space of maximal index [see the preceding review for explanation of terms] is constructed which is composed of s curves with indices $i_1, i_2, \cdots, i_s$, respectively, each of which is the continuous image of a circle. The author shows that a plane curve with equation $f_n(x)-y^2g_{n-2}(x)=0$ satisfies his conditions in the case where $q=2$ if $f_n(x)$ and $g_{n-2}(x)$ are appropriately chosen polynomials of degrees n and $n-2$, respectively. The general result is obtained by a rational mapping of these curves on spaces of higher dimension.

The author also constructs an algebraic curve of maximal index of order $m+n$ in projective q-space which is composed of two curves, of orders m and n, respectively, each of which is a continuous image of a circle. This is accomplished by a rational mapping of one of the above plane curves on a space of higher dimension in the special case where $m=n$. The general case is then derived from this special case by projections, using the results of the paper reviewed above. *D. Derry* (Saskatoon, Sask.).

Myers, S. B. Arcs and geodesics in metric spaces. Trans. Amer. Math. Soc. 57, 217–227 (1945). [MF 12131]

This paper contains the proof and applications of the following fundamental theorem. In a locally compact, almost complete metric space M, any two points which can be joined by a rectifiable arc can be joined by a geodesic arc. This is an extension of an earlier theorem of Menger. An almost complete metric space is defined as follows. Consider a sequence of points P_i (of a metric space M) with the property that for an arbitrary $\epsilon>0$ there exists an N such that, for $i>N$ and $j>N$, P_i can be joined to P_j by an arc of length less than ϵ. If every such sequence converges, M is called almost complete.

A geodesic metric space is then defined as a rectifiably arcwise connected metric space in which distance is identical with the greatest lower bound of arc length. The class of locally compact geodesic metric spaces contains all symmetric Finsler manifolds, in particular, all Riemannian manifolds. A series of results obtained by Hopf and Rinow for Riemannian manifolds is then generalized to locally compact metric spaces; for example, in a locally compact geodesic metric space completeness is equivalent to the Weierstrass-Bolzano theorem. *C. B. Allendoerfer.*

Kosambi, D. D. Parallelism in the tensor analysis of partial differential equations. Bull. Amer. Math. Soc. 51, 293–296 (1945). [MF 12271]

This is a note on the geometry of the system

$$\frac{\partial^{s+1}x^i}{\partial u^{\alpha_1}\cdots\partial u^{\alpha_{s+1}}}+H^i_{\alpha_1\cdots\alpha_{s+1}}\left(u, x, \frac{\partial x^i}{\partial u^\alpha}, \cdots, \frac{\partial^s x^i}{\partial u^{\alpha_1}\cdots\partial u^{\alpha_s}}\right)=0,$$

$$i, j=1, \cdots, n; \alpha, \beta=1, \cdots, m.$$

The author briefly outlines the procedure for forming a complete set of tensor operators and tensor invariants. For this purpose a covariant differentiation process is assumed involving two types of connection parameters $\gamma_{\alpha j}{}^i$ and $\Gamma_{\beta\gamma}{}^\alpha$. His main thesis is that the best intrinsic choice of the γ is to choose those for which the curvature tensor $K^i_{\alpha\beta j}$ vanishes. With this choice the calculation of invariants is simplified, unrestricted parallel displacement of a space vector over an integral variety is always possible, and the γ may be made to vanish over any given integral variety. Presumably there still remains the difficult problem of determining γ and Γ intrinsically in terms of H.

J. L. Vanderslice (College Park, Md.).

NUMERICAL AND GRAPHICAL METHODS

Zinke, O. Beitrag zur geschlossenen Näherungsdarstellung elliptischer Integrale. Z. Angew. Math. Mech. 21, 114–118 (1941). [MF 12150]

Several approximations are given, of which the following is typical:

$$\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-\frac{1}{2}}d\phi \approx \tfrac{1}{2}\pi(1-.5k^2-.15k^4)^{-\frac{1}{2}}.$$

The relative error is said to be less than .005 provided $k<.87$. *W. Feller* (Ithaca, N. Y.).

Bisshopp, K. E. The inverse of a stiffness matrix. Quart. Appl. Math. 3, 82–84 (1945). [MF 12086]

The essential feature of the matrices considered is that among the elements below the main diagonal only those adjacent to it are different from zero. Such matrices can readily be inverted numerically. *W. Feller.*

Hopstein, N. M. Solution of homogeneous linear equations by iteration method. C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 372–375 (1944). [MF 11622]

Consider the system of n homogeneous linear equations in $x_1, \cdots, x_n$ with determinant $D(\lambda)=|c_{ij}|$, $c_{ij}=-a_{ij}$, $i\neq j$, $c_{ii}=\lambda-a_{ii}$, $i, j=1, \cdots, n$, where the a_{ij} are given constants. For $n=4$, the method of the paper would run as follows. Write the equations in the following form: $x_1=1$,

$$x_2'=(a_{21}+a_{23}x_3+a_{24}x_4)/(\lambda-a_{22}),$$
$$x_3'=(a_{31}+a_{32}x_2+a_{34}x_4)/(\lambda-a_{33}),$$
$$x_4'=(a_{41}+a_{42}x_2+a_{43}x_3)/(\lambda-a_{44}),$$
$$\lambda'=a_{11}+a_{12}x_2'+a_{13}x_3'+a_{14}x_4'.$$

Start out now with arbitrary reasonable values x_2, x_3, x_4, λ and use the equations above to compute x_2', x_3', x_4', λ' with $x_1'=1$. Repeat the process. Conditions for convergence of the process are stated (without proof) in terms of the constants a_{ij} and also in terms of the λ's. These conditions are strongly restrictive; the applicability of the method is correspondingly limited. *A. J. Kempner.*

Turton, F. J. On the solution of the numerical simultaneous equations arising in the analysis of redundant structures. J. Roy. Aeronaut. Soc. 49, 104–111 (1945). [MF 12234]

Satterthwaite, F. E. Error control in matrix calculation. Ann. Math. Statistics 15, 373–387 (1944). [MF 11756]

This paper deals with the accumulation of errors, such as those due to rounding, during certain numerical operations with matrices. By the "norm" of a matrix is meant the square root of the sum of the squares of the matrix elements. It is shown that, if the norm of the matrix $A-I$ is less than 0.35, operations leading to the inverse of A and multiplication by it will be in a state of error control for "Doolittle" methods of calculation. *T. E. Sterne.*

Wayland, Harold. Expansion of determinantal equations into polynomial form. Quart. Appl. Math. 2, 277–306 (1945). [MF 11771]

This paper gives an expository account of the various known methods for expanding a determinantal equation

$$|A_0\lambda^n+A_1\lambda^{n-1}+\cdots+A_n|=0,$$

where the A_j are square matrices of order m. First, for the equation $|A-I\lambda|=0$, the methods due to Leverrier, Krylov, Danielewsky, Reiersøl and Samuelson are outlined and illustrated by examples. Secondly, for the equation $|A-B\lambda|=0$, where B is a nonsingular matrix, the method of reciprocation leading to the equation $|B^{-1}A-I\lambda|=0$ and Masuyama's modification of Danielewsky's method are discussed. Finally, for the general equation

$$|A_0\lambda^n+A_1\lambda^{n-1}+\cdots+A_n|=0,$$

the methods described are the following: (1) transformation to the form $|A-I\lambda|=0$ involving determinants of order mn; (2) interpolation-formula method, evaluating the determinant for $nm+1$ values of λ by the Gregory-Newton formula; (3) the method of undetermined coefficients. The paper closes with formulas and tables which give the number of multiplication-division and addition-subtraction operations involved in each method for various values of m, in order to guide a computer as to the best choice of method for his particular problem. *M. Marden.*

Eriksson, H. Adolf S. A technique for the approximate calculation of eigen-values as zeros of a determinant. Application to the Li⁺-ion in the ground state. Ark. Mat. Astr. Fys. 30B, no. 6, 8 pp. (1944). [MF 12008]

The author considers the use of the Ritz variation method for the solution of the problem $L_\epsilon(U)=0$, where L_ϵ is a linear operator which is a function of a parameter ϵ, the eigenvalue. If the variational parameters are linear, $U=\sum c_nf_n$, the problem reduces to solving for ϵ the secular equation $|(f_m, L_\epsilon f_n)|=0$. The author's procedure consists in obtaining from $\{f_n\}$ new functions $\{\varphi_n\}$ which form an orthogonal set with respect to the operator L_ϵ; that is, $(\varphi_n, L_\epsilon\varphi_m)=0$ when $n\neq m$. This can be performed by means of the usual recursion procedure, which leads to a set of recursion formulas for the matrix elements $(\varphi_n, L_\epsilon\varphi_n)$ and $(f_m, L_\epsilon\varphi_n)$. The approximate equation obtained by replacing the exact secular determinant by its first n rows and columns is then equivalent to $(\varphi_n, L_\epsilon\varphi_n)=0$. This equation then yields the nth approximation for ϵ; the nth approximation for U is the corresponding φ_n. As an example, the author applies the method to computing the ground state of the Li^+-ion, a two electron problem. The procedure is shown to involve n^2 tabulated steps for the evaluation of an nth order determinant. *H. Feshbach.*

Reiz, Anders. On the numerical solution of certain types of integral equations. Ark. Mat. Astr. Fys. 29A, no. 29, 21 pp. (1943). [MF 12022]

A frequently used method of numerical integration of integral equations of the second kind consists in replacing the integrals by Riemann sums and thus reducing the problem to a system of algebraic linear equations. The author shows that this method can be applied also in the case of integral equations over an infinite domain and discusses the appropriate spacing of the points. Several examples are worked out. *W. Feller.*

Collatz, L. Graphische Lösung von Randwertproblemen bei gewöhnlichen linearen Differentialgleichungen 2. Ordnung. Z. Angew. Math. Mech. 23, 237–239 (1943). [MF 11736]

The differential equation

$$y''+r(x)y'+q(x)y=f(x)$$

with boundary conditions $y(a)=\alpha$, $y(b)=\beta$ is transformed to a finite difference equation with constant interval h. It is shown that, if several solutions of this difference equation, satisfying the first (but not in general the second) boundary condition, are plotted, the chords through the two points (x_i, y_i), (x_{i+1}, y_{i+1}) for different solutions but for the same i all pass through a fixed point S_i. A method of finding the S_i is given, after which the construction of the particular solution satisfying the second boundary condition is a simple matter. *W. E. Milne* (Corvallis, Ore.).

Allen, D. N. de G., Southwell, R. V., and Vaisey, Gillian. Relaxation methods applied to engineering problems. XI. Problems governed by the "quasi-plane-potential equation." Proc. Roy. Soc. London. Ser. A. 183, 258–283 (1945). [MF 12048]

[Part X appeared in the same Proc. Ser. A. 183, 125–134 (1944); these Rev. 6, 137.]

A detailed exposition of the finite difference method applied to the numerical solution of boundary value problems connected with the "quasi-plane-potential" equation $(\phi u_x)_x+(\phi u_y)_y+Z=0$. The authors show that the methods developed for the harmonic case apply almost without change. Some improvements in the technique are suggested but they cannot be described in a few words.

W. Feller (Ithaca, N. Y.).

Schmidt, Kurt. Über die experimentelle Lösung ebener Potentialaufgaben durch elektrische Dipolfelder. Ing.-Arch. 14, 30–52 (1943). [MF 11430]

A high precision electrolytic tank for the solution of two-dimensional potential problems is described, and the procedure is illustrated for examples of ideal fluid flow around a cylinder (with or without circulation) and for the flow around a plate with free rays; in the former example the flow is studied inside the cylinder where the dipole singularity at the center is replaced by two nearby electrodes of opposite polarity. The flow around the plate is reduced to a semicircular region in the velocity (hodograph) plane. *H. Poritsky* (Schenectady, N. Y.).

Concordia, C. Network- and differential-analyzer solution of torsional oscillation problems involving nonlinear springs. J. Appl. Mech. 12, A-43–A-47 (1945). [MF 12045]

This paper (a) points out that, just as in the case of linear systems, the "alternating-current network analyzer" may be used to extend the graphical method to the direct solution of torsional-oscillation problems involving many masses and springs, more than one of which may be nonlinear, and (b) illustrates the use of the "differential analyzer" in solving steady-state vibration problems in which the approximate graphical method cannot be applied or does not give satisfactory solutions.

From the author's summary.

Kron, Gabriel. Electric circuit models of the Schrödinger equation. Phys. Rev. (2) 67, 39–43 (1945). [MF 11876]

This is a continuation of work undertaken by G. K. Carter and the author [Kron, J. Appl. Mech. 11, A-149–A-161 (1944); Carter and Kron, J. Franklin Inst. 238, 443–452 (1944); these Rev. 6, 140]. Equivalent circuits are now developed to represent the Schrödinger equation for one, two and three independent space variables in orthogonal curvilinear coordinate systems. The networks account for the presence of an arbitrary potential function in the Schrödinger equation and solutions to these network problems may be obtained on the A.C. network analyzer, or by other known methods in circuit theory.

A. E. Heins (Cambridge, Mass.).

Carter, G. K., and Kron, Gabriel. A. C. network analyzer study of the Schrödinger equation. Phys. Rev. (2) 67, 44–49 (1945). [MF 11877]

Measurements are carried out on the circuits described by Kron in the paper reviewed above. Eigenfunctions and eigenvalues are determined by these circuits for several particular problems including the linear harmonic oscillator, potential well, rigid rotator, etc. *A. E. Heins.*

Kron, Gabriel. Numerical solution of ordinary and partial differential equations by means of equivalent circuits. J. Appl. Phys. 16, 172–186 (1945).

Numerical methods are developed to solve certain classes of partial differential equations with the aid of equivalent networks. Initial value, boundary value and characteristic value problems are considered. *A. E. Heins.*

Meister, F. J. Anwendung von elektrischen Verstärkern für Integrationszwecke. Forschung Gebiete Ingenieurwesens. Ausg. B. 14, 124–131 (1943). [MF 11852]

A circuit is described for performing one or two indefinite integrations of a function. The integration is done electrically with a resistance condenser circuit and amplifier, the flat pentode characteristic being used to obtain a high RC without requiring excessive gain. The curve to be integrated is recorded on film as a variable area and taken off by slit scanning. Both input and output are recorded simultaneously on an oscillograph. Some examples of integrated curves are shown. *C. E. Shannon.*

Fürth, R., and Pringle, R. W. A new photo-electric method for Fourier synthesis and analysis. Philos. Mag. (7) 35, 643–656 (1944). [MF 11954]

Synthesis is accomplished by a photo-electric method. A disc on which variable-area harmonic waves have been photographed is rotated between a projector and a set of optical wedges, the latter controlling the amplitudes of nine sine and cosine components. All the resulting light beams are brought to focus on a photo-cell, the output of which is amplified and applied to the vertical deflection plates of a cathode-ray oscilloscope. The authors have built only a demonstration model and the results given are only qualitative.

Using a similar optical system, the authors provide for the introduction of another variable area pattern on the rotating disc, to produce a photo-cell voltage varying in accordance with a function to be analyzed. This voltage, together with one of the sinusoidal voltages, is applied to two grids of a "mixer" tube. By reading the time average of the increment of plate current they obtain a measure of the integral

$$T^{-1}\int_0^T e_1(t)e_2(t)dt.$$

One of these voltages, say $e_1(t)$, is the function to be analyzed. The other is a sine or cosine of appropriate frequency.

The synthesizer as described has some appeal as a demonstration apparatus but can hardly be considered adequate for precision work without further development. The

analyzer is not impressive in concept, its operation is slow and uneconomic, and the only results given are poor.
S. H. Caldwell (Cambridge, Mass.).

Grützmacher, Martin. Eine neue Darstellungsform der harmonischen Analyse und ein neuer mechanischer Kurvenanalysator. Akustische Z. **8**, 49–63 (1943). [MF 10252]

Kallenbach, Werner. Bemerkung zu der vorstehenden Arbeit von M. Grützmacher über ein neues Analysierverfahren. Akustische Z. **8**, 63–65 (1943). [MF 10253]

The first paper describes a mechanism for constructing the curve

$$x_n(t)=\pi^{-1}\int_0^t f(\tau)\cos n\tau d\tau,\quad y_n(t)=\pi^{-1}\int_0^t f(\tau)\sin n\tau d\tau$$

($0\leqq t\leqq\pi$), for which the coordinates of the end point are the coefficients a_n and b_n in the Fourier expansion of $f(t)$. The device makes use of a property of this curve proved in the second paper, namely that the radius of curvature is proportional to $f(t)$. It is comparatively simple to construct, and is said by the author to be simpler and quicker in operation than harmonic analysers of conventional types. In the analysis of test functions, a precision of about 1% in amplitude and n degrees in phase of the harmonics was obtained up to $n=12$. Stepwise construction of the curve without aid from the machine is also discussed.
R. L. Dietzold (New York, N. Y.).

Somerville, J. M. Harmonic synthesizer for demonstrating and studying complex wave forms. J. Sci. Instruments **21**, 174–177 (1944). [MF 11370]

This paper describes an electronic device used in connection with a cathode ray oscillograph to exhibit the wave forms available from seven harmonically related sinusoids, each of which may be easily adjusted in amplitude and phase. *R. L. Dietzold* (New York, N. Y.).

Shilton, A. A machine for harmonic synthesis. Proc. Phys. Soc. **56**, 130–132 (1944). [MF 10358]

This machine is intended for classroom demonstration of the addition of complex waves. It utilizes hydraulic means for summing the motions of three pistons which are driven by a system of gears and cams from a main shaft. Some choice of harmonics and some control over their amplitude and phase are possible by changing the gears and cams.
R. L. Dietzold (New York, N. Y.).

Meyer zur Capellen, Walther. Das Reibradgetriebe als Integrator. Z. Instrumentenkunde **63**, 241–258 (1943). [MF 10083]

The author states that the principal objections to the Bush differential analyzer have been removed in the "Funktionsrechner," designed in 1937 by Engel and Taskowski. The only modifications mentioned are the substitution of electrical follow-ups for mechanical torque amplifiers and the replacement of drawing boards by drums for the recording and tracking of functions. Otherwise the device follows closely the pattern of the early differential analyzers, utilizing disk and wheel integrators and fully mechanical couplings. Connection diagrams are given for the solution of many typical differential equations, with close attention to the adjustment of scale factor.
R. L. Dietzold (New York, N. Y.).

Kormes, Jennie P., and Kormes, Mark. Numerical solution of initial value problems by means of punched-card machines. Rev. Sci. Instruments **16**, 7–9 (1945). [MF 11874]

Benjamin, Kurt. An I.B.M. technique for the computation of $\sum X^2$ and $\sum XY$. Psychometrika **10**, 61–67 (1945). [MF 12142]

Meyer zur Capellen, Walther. Übertragungs- und Wendegetriebe bei der "Monroe"-Rechenmaschine. Z. Instrumentenkunde **63**, 316–322 (1943). [MF 12094]

Müller, Max. Abschätzung des Fehlers bei der Tangentennäherungskonstruktion von Pirani. Math. Z. **49**, 380–388 (1944). [MF 11997]

Numerical estimates of the error in Pirani's construction which approximates the tangent t of a curve C at the point P by the chord joining the two intersections of C with a circle about P. *W. Feller* (Ithaca, N. Y.).

Hansel, C. W. Graphical computation. Philos. Mag. (7) **35**, 159–169 (1944). [MF 10865]

A plea for a more extensive utilization of graphical charts in science and engineering. The author takes issue with widely held views that graphical methods must give way to numerical methods when high accuracy is required. He shows in detail, with the aid of an elaborate series of charts, how graphical methods may be made to comply with highly accurate data. *H. Blumberg* (Columbus, Ohio).

Knobloch, Hans. Über eine Verzerrungsfunktion in der Nomographie. Z. Angew. Math. Mech. **21**, 103–107 (1941). [MF 12148]

Let $x=x(a)$ be a function of a variable a which is represented by a scale on a straight line. The distortion of this scale is defined as $V_{x(a)}=\vartheta_a x''(a)/x'(a)$, where ϑ_a is a constant increment on the a-scale. The author considers adjacent scales representing monotonic functions. He shows that it is possible to divide the distortion between the two scales so that the distortion of each scale is less than the total distortion. *E. Lukacs* (Berea, Ky.).

MATHEMATICAL PHYSICS

Sobolev, V. V. Point source of light between parallel planes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 172–173 (1944). [MF 11632]

An explicit solution of the following problem is given. A point source of light is placed between two parallel planes which are assumed to have perfect diffuse reflectance. The intensity of the light source is independent of the direction and the medium between the planes is perfectly transparent. The problem is to find the illumination $E_1(r)$ and $E_2(r)$ of the two planes. These functions satisfy a system of two simultaneous integral equations which are solved by the method of Fourier's integral. It is noted that the same method may be applied in the more general case where the radiation of the light source is symmetrical with respect to the line drawn through it perpendicular to the planes and the medium between the planes is absorbing but not scattering. *R. K. Luneberg* (Buffalo, N. Y.).

Lifshitz, I. M. Optical behaviour of non-ideal crystal lattices in the infra-red. II. Acad. Sci. USSR. J. Phys. 7, 249–261 (1943). [MF 11000]

[Part I appeared in the same J. 7, 215–228 (1943); these Rev. 6, 112.] In this paper a calculation is made of the absorption due to the existence of isotopes. In the equation for normalized displacements

$$(A-z+\Lambda)u_0=\phi,\quad \Lambda_{kk'}^{ss'}=-\delta_{kk'}\delta_{ss'}(\mu_s\omega^2/m_s)\gamma_k{}^s,$$

the elements of the diagonal matrix Λ are supposed to be small ($\mu_s/m_s \ll 1$) so that a solution can be found by expanding in powers of Λ. In the analytical work it is necessary to study an integral of the form

$$L(z)=(1/V^*)\int h(t)dt/[s(t)-z],\qquad z=\omega^2+i\gamma,$$

as $\gamma\to 0$, $h(t)$ and $s(t)$ being real periodic functions of t having the symmetry of the reciprocal lattice. The integral is taken over the volume of the elementary cell of the reciprocal lattice but for the sake of simplicity an examination is first made of the one-dimensional case in which the integral is a simple integral. After a study of the solution the results are discussed. The absorption coefficient is found to be proportional to the product of the concentrations $c_s(1-c_s)$ and to the square of the ratio of the difference of the masses of the isotope to the mean mass, that is, to $(\mu_s/m_s)^2$. At the edges of the absorption band (the interval of eigenfrequencies) the absorption coefficient vanishes as $(z-z_m)^{\frac{1}{2}}$. *H. Bateman* (Pasadena, Calif.).

Foulkes, P. Extension of Maxwell's equations. J. Proc. Roy. Soc. New South Wales 78, 14–16 (1944). [MF 11124]

The author generalizes Maxwell's equations to include the existence of magnetic currents as recently claimed by Ehrenhaft. Under the circumstances, as previously pointed out by Whittaker [Proc. Roy. Soc. Edinburgh 46, 116–125 (1926)], the equations become even more symmetrical than the standard Maxwell equations. *H. Poritsky.*

Watson, W. H. A theory of the creation of electric charge. Canadian J. Research. Sect. A. 23, 33–38 (1945). [MF 12052]

This paper deals with a modification of Maxwell's equations for electromagnetic fields by the addition of terms which cause creation and destruction of electric charge. The modified equations are

$$\mathbf{j}=\operatorname{curl}\mathbf{H}-\partial\mathbf{D}/\partial t+(1/\mu_0)\operatorname{grad}N,$$
$$\rho=\operatorname{div}\mathbf{D}-\kappa_0\partial N/\partial t;$$

the other two equations are unchanged:

$$\operatorname{curl}\mathbf{E}=-\partial\mathbf{B}/\partial t,\quad \operatorname{div}\mathbf{B}=0.$$

The latter equations permit the introduction of the vector and the scalar potentials in the usual way, but the equation of continuity for the charge and current densities now takes the form

$$\mu_0(\partial\rho/\partial t+\operatorname{div}\mathbf{j})=\square N,$$

and the potentials must satisfy the condition

$$\operatorname{div}\mathbf{a}+(1/c^2)\partial\varphi/\partial t=-N.$$

The equations of motion of charged particles in this theory are derived, and the possibilities of their creation and annihilation are discussed. It is suggested that the electromagnetic potentials be given absolute values such that the magnitude of the four-vector at the place where the charge is created determines the mass of the particle by the relation

$$mc=e(\varphi_0^2/c^2-\mathbf{a}_0^2)^{\frac{1}{2}}.$$

Possible modifications of quantum electrodynamics are also discussed. *S. Kusaka* (Northampton, Mass.).

Watson, W. H. Discontinuous motion of an electric particle. Canadian J. Research. Sect. A. 23, 39–46 (1945). [MF 12053]

The assumption is made that the world line of a particle consists of a discrete series of regularly spaced events instead of a one-dimensional continuum as in the usual theories. It is assumed that electric charge is alternately created and destroyed at regular intervals, and the formalism developed by the author for such circumstances is used [see the preceding review]. It is shown that waves of electric potential, to which are assigned mass, energy and momentum, are set up in the vicinity of the light cone, and that it divides space-time into two regions, one in which the field is static and another in which it vanishes. The electromagnetic field singularities are interpreted as boundaries of the field, which may be considered to be propagated like electromagnetic waves in a wave guide. The connection of these ideas with quantum theory is briefly discussed. *S. Kusaka* (Northampton, Mass.).

Feld, J. N. Reciprocation theorem of electrodynamics for transitory processes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 280–283 (1943). [MF 11070]

In this paper the author establishes for transient fields a reciprocation theorem analogous to the Sommerfeld theorem for harmonic fields. This is done by utilizing the operational transforms of the actual field components. The result obtained is

$$\int_0^t p_1'(\tau)E_2(t-\tau)d\tau=\int_0^t p_2'(\tau)E_1(t-\tau)d\tau,$$

where p_1, p_2 are the moments of two dipoles and E_1 the field at the second dipole due to the first one, E_2 the field at the first due to the second. *H. Poritsky.*

Aharoni, J. Effect of space charge on electron beams. Philos. Mag. (7) 35, 36–50 (1944). [MF 10864]

The Maxwell equations and the "fluid flow equations" of electronic flow are solved by assuming that the velocity components are linear in the coordinates in a plane normal to the beam axis. *H. Poritsky* (Schenectady, N. Y.).

Jachnow, Walter. Theoretische Untersuchungen über Strahlungsdiagramm und Strahlungswiderstand bei fortschreitenden Wellen verschiedener Phasengeschwindigkeit. Elektr. Nachr. Techn. 19, 147–155 (1942). [MF 11357]

The polar radiation diagram and the radiation resistance R_s of a thin linear straight aerial of length l are computed under the assumption that the ratio $\beta=v/c$ of phase velocity along the wire to that in free space is varied. (This condition can be realized approximately by a Beverage antenna with equal nonradiating phasing networks interspersed periodically at distances which are not large compared with $\lambda/4$). The computations are based on the zone $r\gg\lambda$ and are conventional and straightforward, there being no corrections on account of the assumption of negligible wire thickness. It appears that the aerial works quite differently

for β appreciably less than and greater than 1. In the first case, the lobes are comparatively wide and tend to cover the whole orbit fairly equally for large l/λ; R_s increases to a peak at $l/\lambda=\frac{1}{2}$ and reaches an asymptotic value with minor irregular oscillations when l/λ is larger than 1. For $\beta>1$ the lobes narrow rapidly with increasing l, and when l/λ is larger than $\frac{3}{2}$ the main lobe with the angle $\theta=\cos^{-1}\beta$ becomes predominant; the radiation resistance increases asymptotically in linear fashion. Conditions change rapidly for variation of β near $\beta=1$ if $l/\lambda\gg\frac{1}{2}$. *H. G. Baerwald.*

Malov, N. On the calculation of the radiation field of a wave conductor. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 14, 224–225 (1944). (Russian) [MF 12223]

On the basis of the law of the conservation of energy it is shown that the generally accepted method for the calculation of the radiation field of a wave conductor is invalid for waves whose lengths are near the "critical value." The critical value of a wavelength for a given conductor is the length of the longest wave which can be propagated in the conductor and is approximately the length of the conductor. The method of proof consists in showing that a certain constant (which must have the value unity on the basis of the law of the conservation of energy), when calculated by the generally accepted methods for wavelengths near the critical value, differs considerably from its prescribed value. *H. P. Thielman* (Ames, Iowa).

King, Ronold, and Harrison, Charles W., Jr. Mutual and self-impedance for coupled antennas. J. Appl. Phys. 15, 481–495 (1944).

The authors discuss the currents and impedances in a system of two parallel identical straight wire antennas of half-length h and radius a separated by distance b. Integral equations for the currents in the antennas are obtained and solved by Hallén's method of successive approximations in terms of the parameter $\Omega=2\log(2h)/a$. Curves of the self-impedance, the mutual impedance and the input impedance as functions of the separation b/λ are drawn in each of four cases for the values $h=\lambda/4$, $\lambda/2$ and $\Omega=\infty$, 30, 20, 10. Only the first order terms in $1/\Omega$ are used, though it is realized that succeeding terms are not negligible unless Ω is very large. It is emphasized that the self-impedance of an antenna when in the presence of another antenna differs from its self-impedance when isolated unless the antenna is infinitely thin. *M. C. Gray* (New York, N. Y.).

Aharoni, J. A general theory of antennae. Philos. Mag. (7) 35, 427–459 (1944). [MF 11298]

The author first shows how the usual circuit theory may be derived from Maxwell's equations assuming that all dimensions are small compared with the wave length and that radiation may be neglected. For systems which radiate electromagnetic energy an analogous method is then used to obtain an integral equation for the current density assuming only that the lines of current flow are stationary. For small loops and short thin wires this equation can be used to determine the general form of the (complex) inductances and capacitances of the corresponding circuit representations. The integral equation includes Hallén's equation for a straight wire antenna as a special case and the author includes an outline of Hallén's solution. The application of the theory to a receiving antenna is also discussed. *M. C. Gray* (New York, N. Y.).

Schelkunoff, S. A. Impedance concept in wave guides. Quart. Appl. Math. 2, 1–15 (1944). [MF 10332]

The author's viewpoint is that the impedance concept is such a powerful tool in transmission theory that its introduction in wave guide theory is highly desirable, especially if the wave guides are to be used as transmission systems. The present paper is mainly expository and discusses the general properties of the impedances which may be associated with each propagation mode in a uniform wave guide. The author shows how discontinuities in such guides may be represented by appropriate impedances. Complex discontinuities in wave guides may be analyzed into simpler discontinuities and represented by appropriate impedances, at least as far as the transmission properties of the guide are concerned. *M. C. Gray* (New York, N. Y.).

Fränz, Kurt. Eine Verallgemeinerung des Fosterschen Reaktanztheorems auf beliebige Impedanzen. Elektr. Nachr. Techn. 20, 113–115 (1943). [MF 12095]

On the basis of an analytic representation of the general network function, limits are deduced for the simulation of a general impedance by means of resistive networks. *E. Weber* (Brooklyn, N. Y.).

Vlasov

Blasov, A. A. Generalization of the concept of electronic plasma. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] 8, 248–266 (1944). (Russian) [MF 11951]

This is a rather general discussion of the connection between the compounded "large distance" interaction effects in an electron gas and the macro-concepts of the gaseous, liquid and crystalline states, etc. The author starts from the general equation of state

$$-\partial f/\partial t=[Hf]+[\partial f/\partial t]^{st},$$

where $f(x, y, z, \xi, \eta, \zeta, t)$ is the distribution function; $K(|\mathbf{r}-\mathbf{r}'|)$ is the potential energy of central force particle interaction; H is the Hamiltonian

$$H(x, y, z, p_x, p_y, p_z)=\sum_{i=1}^{3} p_i^2/2m+\int K(|\mathbf{r}-\mathbf{r}'|)\int f(\mathbf{r}, \mathbf{v}, t)d\tau' d\omega,$$

$d\tau=dxdydz$, $d\omega=d\xi d\eta d\zeta$; $[Hf]=\sum_{i=1}^{3}\{f_{x_i}H_{p_i}-f_{p_i}H_{x_i}\}$ and $[\partial f/\partial t]^{st}$ is the change of the distribution function due to the "close range" interactions between neighbors. He derives the following simplified approximate equation which, under certain order-of-magnitude assumptions, will adequately describe the "large distance" cooperative phenomena in the electron gas:

$$\partial f/\partial t+\mathbf{v}\operatorname{grad}_r f-m^{-1}\operatorname{grad}_r V\cdot\operatorname{grad}_v f=0,$$

where

$$V(\mathbf{r}, t)=\int K(|\mathbf{r}-\mathbf{r}'|)\int f(\mathbf{r}, \mathbf{v}, t)d\tau' d\omega.$$

This equation is nonlinear. Under certain reasonable assumptions, the phenomena in question involve only the consideration of the behavior of the solution in the neighborhood of some "stationary" solution $f_0(\xi, \eta, \zeta)$ corresponding to constant density and thus involving the velocities, but independent of space and time. Putting, accordingly, $f=f_0+\varphi(x, y, z, \xi, \eta, \zeta, t)$, the author obtains the linearized equation of state

$$\partial\varphi/\partial t+\mathbf{v}\operatorname{grad}_r\varphi=m^{-1}\operatorname{grad}_v f_0\operatorname{grad} r\int\int K(|\mathbf{r}-\mathbf{r}'|)\varphi(\mathbf{r}, \mathbf{v}, t)d\omega d\tau',$$

which forms the point of departure. Of the two possible types of solution, one involving and one not involving changes of density of the medium, only the first is of importance for the present purpose. This solution is obtained in "wave" form $\varphi \sim g_{\mathbf{k}}(\xi, \eta, \zeta) \exp(i\omega t - i\mathbf{kr})$, the "amplitude" distribution satisfying the integral equation

$$g_{\mathbf{k}}(\xi, \eta, \zeta) = \frac{\sigma(k)}{m} \frac{(\mathbf{k}\nabla)f_0}{[\mathbf{kv}-\omega]} \int_{-\infty}^{\infty} g_{\mathbf{k}}(\xi, \eta, \zeta)\, d\omega,$$

where $\sigma(k) = 4\pi \int_0^\infty K(\rho)(\rho/k) \sin k\rho\, d\rho$, with the condition

$$\frac{\sigma(k)}{m} \int_{-\infty}^{\infty} \frac{(\mathbf{k}\nabla)f_0}{[\mathbf{kv}-\omega]} d\xi d\eta d\zeta = 1,$$

which represents the dispersion law of the waves. A detailed discussion of the general solution as well as of various more specialized cases, with the appropriate physical interpretations, forms the bulk of the paper. The corresponding chapter headings are: Dispersion law of the waves for arbitrary central forces and arbitrary starting-off distribution functions f_0; Existence of maximal frequencies and wave numbers for arbitrary central forces and starting-off distribution functions; Vibration spectrum of an electron gas; The spontaneous excitation of oscillations for sufficiently fast rates of decay of the repulsion forces; Genesis and spontaneous formation of crystal structure.

H. G. Baerwald (Cleveland, Ohio).

Frenkel, J. I. To the theory of seismic and seismo-electrical phenomena in humid soils. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **8**, 133–150 (1944). (Russian. English summary) [MF 12181]

The moist ground is treated as a two-phase system, constituted by a porous solid body which is permeated by a liquid. The equations of equilibrium and motion are established. They can be specified by two independent stress tensors, one referring to the solid skeleton and the other to the liquid phase, the deformations (strains) in each phase being, however, functions of the stresses in both. These tensors are given for the solid part by

$$\phi_i^{(1)} = \sum_k \partial T_{ik}/\partial x_k + (1-f)K_2 \partial\varphi/\partial x_i,$$

and for the liquid by $\phi^{(2)} = fK_2 \nabla\varphi$.

Here the T_{ik} are the components of the stress tensor in the solid, f is the coefficient of porosity, K_2 the modulus of compressibility of the liquid and φ the relative change in volume of the liquid. Darcy's law, which connects the flow of liquid with the resulting friction forces, is modified to account for the compressibility of the solid and for the forces of inertia arising in the case of a nonsteady motion. The equations of motion are applied in a linearized form to the seismic vibrations propagated in a moist ground in the form of longitudinal and transversal waves of small amplitude. The solution of these equations by successive approximations is discussed and the first approximation is worked out in detail. The velocity of propagation and the damping coefficient are obtained as functions of the various parameters specifying the structure and mechanical properties of the ground. The theory is applied to the calculation of the intensity of electrical vibrations which are associated with the mechanical ones as a result of the existence of a relative velocity between the solid phase and the water, in connection with the presence of an electro-kinetic potential jump on their surface of contact. This leads to a refinement of results obtained in a rather qualitative form by Ivanov, who discovered the existence of the seismo-electric effect under consideration.

It is stated that solid metallic bodies can be treated as a two-phase system in a manner analogous to the method used in this paper. In such bodies the rôle of the solid skeleton is played by the crystalline framework consisting of positive ions, while the part of the liquid phase is taken by the "electric fluid" consisting of the collection of free electrons. It follows that the propagation of longitudinal sound waves in metallic bodies is accompanied by electrical effects similar to those which are observed in connection with the propagation of seismic waves. This question is to be treated in another paper.

H. P. Thielman.

Kogbetliantz, E. G. Quantitative interpretation of maps of magnetic and gravitational anomalies by mathematical methods. Quart. Appl. Math. **3**, 55–75 (1945). [MF 12085]

This paper deals with the mathematical determination of quantities characterizing an underground structure such as an ore body or an oil deposit. The method presented can be applied to structures much smaller than those which can be treated by previous methods. Maps are prepared showing the empirical variation of the gravitational forces on the surface of the earth above a structure. Equations are derived involving the moments of various orders of the functions representing this variation. The moments are computed from the data on the maps, and the equations are solved by an iteration process for the quantities characterizing the structure. A similar procedure can be carried out using maps showing the empirical variation of the magnetic intensity. The method is applied to a structure with a uniform vertical cross section (a two-dimensional problem) and to a structure with a vertical axis of symmetry.

G. E. Hay (Providence, R. I.).

Charny, I. A. Flow of oil to wells in reservoirs of circular and band-like form. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 156–159 (1944). [MF 11630]

Charny, I. A. Flow of oil to wells in reservoirs of oval and crescent form. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 205–207 (1944). [MF 11635]

The first paper concerns the use of conformal mapping on the basis of Dupuis' formula for radial flow to a single well located at the center of an oil-reservoir of circular form. It studies the production of a well placed either eccentrically or on the axis of an oil producing stratum pinched out at a given angle (circular sector) and the flow to and the production of an infinite array of equidistant wells running parallel to the boundary of an oil bearing layer having the form of a half-plane. The second paper applies the same method to the production problem of a well in the case of oil-reservoirs of oval form.

E. G. Kogbetliantz.

Buckley, Floyd. Transformations of the fundamental equations of thermodynamics. J. Research Nat. Bur. Standards **33**, 213–233 (1944). [MF 11390]

A substitution group for generating families of thermodynamic formulas (involving the various potentials, etc.) is derived. The method of derivation is based upon the transformation properties of a "group of functions" (in the sense of Lie) under contact transformation. There exists a characteristic function and a "group of functions" for each

representation, and to each function of the group there is an associated contact transformation which transforms the group into its equivalent in another representation. These ideas are applied systematically to the study of various forms into which one can put the thermodynamic equations of a physical system. *B. O. Koopman.*

Beck, Guido. Introduction à la théorie des quanta. I. Le problème de la physique théorique. V. La mécanique quantique. Revista Fac. Ci. Univ. Coimbra 10, 91 pp. (1942). [MF 12417]

de Broglie, Louis, et Tonnelat, Marie-Antoinette. L'introduction des constantes de Coulomb et de Newton en mécanique ondulatoire. C. R. Acad. Sci. Paris **218**, 373–376 (1944). [MF 12108]

Kwal, Bernard. Sur la mécanique ondulatoire des corpuscules élémentaires. C. R. Acad. Sci. Paris **218**, 548–550 (1944). [MF 12123]

Chang, T. S. A note on the Hamiltonian theory of quantization. Proc. Roy. Soc. London. Ser. A. **183**, 316–328 (1945). [MF 12050]

A general method is developed for quantizing fields which satisfy, in addition to the field equations, a set of auxiliary conditions. Assuming the existence of a Lagrangian which is a function of the independent variables, the dependent field variables and their first derivatives, and a set of conditions involving these same variables, the variation of the Lagrangian is carried out subject to the auxiliary conditions. From the equations obtained in this way, conjugate variables and the Hamiltonian function are defined and the equations expressed in canonical form. The quantization is then carried out in the usual way. It is pointed out that for fields with some conjugate variables missing, it is sometimes possible to introduce some auxiliary conditions without affecting the field equations but with the result that all the new canonical variables are present, thus allowing the application of the standard method of quantization. To illustrate this method the Maxwell electromagnetic field is quantized subject to the usual Lorentz condition. The supplementary condition on the wave functional, $(\nabla\cdot A+c^{-1}\partial\phi/\partial t)\psi=0$, is found to be shifted to other conditions on ψ. It has the advantage of having all the conjugate variables and of not putting the time component of the field on a different footing, but the practical improvement, if any, is small. *S. Kusaka.*

Foldy, Leslie L. The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers. Phys. Rev. (2) **67**, 107–119 (1945). [MF 11966]

A collection of n point scatterers has the distribution function $f(r_1, s_1; \cdots; r_n, s_n)$, where s_i is a parameter associated with the ith scatterer. Let

$$E(r, r_j)=(\exp(-ik_0|r-r_j|))/|r-r_j|.$$

The wave function, assuming spherically symmetric scattering proportional to the external field, is then

$$(1)\qquad \psi(r)=\psi_0(r)+\sum g_j\psi^i(r_j)E(r, r_j),$$

where $\psi_0(r)$ is the incident wave and g_i depends on s_i and the frequency, while $\psi^i(r_j)$ is the external field at the jth scatterer. Multiply (1) by f and integrate over $\{r_j\}$, $\{s_j\}$ values. This gives the mean of $\psi(r)$. On the right side there occurs the mean of $\psi^i(r_j)$ over $\{r_i\}$ values excepting r_j. The evaluation of this last mean is not feasible, so the author introduces the approximation of replacing it by the mean of $\psi(r_j)$. Using bars for means we then have

$$(2)\qquad \overline{\psi(r)}=\psi_0(r)+\int_V G(r')E(r, r')\overline{\psi(r')}dr'.$$

This is a Fredholm type of equation for the mean of $\psi(r)$. A similar equation is found for the mean of $|\psi(r)|^2$. This involves the same sort of approximation as that mentioned above, but here even the physical basis is weak. The writer mentions the Neumann expansion for the solution of (2) and gives physical interpretations for the terms.

D. G. Bourgin (Urbana, Ill.).

Havas, Peter. On the interaction of radiation and two electrons. Phys. Rev. (2) **66**, 69–76 (1944). [MF 10931]

In the quantum theory of radiation, the effect of retardation in the interaction of two electrons in the presence of radiation is ascribed to the mutual emission and absorption of virtual light quanta. The author uses this theory with the usual first-order perturbation method to obtain a formula for the retarded interaction of two electrons. This formula, which is obtained by summing over all intermediate states involving virtual quanta, has the same form regardless of the number of virtual quanta required. The formulas of Møller for one light quantum and no light quantum are obtained as special cases. *O. Frink.*

Hsüeh, C. F., and Ma, S. T. Approximate solutions of the integral equations in scattering problems. Phys. Rev. (2) **67**, 303–307 (1945). [MF 12446]

The authors consider the solution of the integral equation

$$(1)\qquad U_f=H_{fi}-i\pi\sum_{\nu_{f'}}\int H_{ff'}U_{f'}\rho_{f'}d\Omega_{f'}$$

by a variational procedure. This equation occurs in the recently developed quantum theory of scattering [for example, see Heitler, Proc. Cambridge Philos. Soc. **37**, 291–300 (1941); Wilson, same Proc. **37**, 301–316 (1941); these Rev. **4**, 95]. In that theory, the probability amplitude for the final state f is related to U_f by

$$a_f=U_f\frac{\exp\{-\frac{1}{2}\Gamma t+i(E_f-E_i)t/\hbar\}-1}{E_i-E_f-\frac{1}{2}i\hbar\Gamma},$$

where E_i is the energy of the initial state; E_f, that of the final state; and $a_i=e^{-\frac{1}{2}\Gamma t}$. The quantities H_{fi} are matrix elements of the perturbing Hamiltonian between the indicated states f and i. Finally, $\rho_{f'}$ is the density per unit energy range per unit solid angle of the states f'.

The variational equation replacing (1) is

$$\sum_{\nu_i}\sum_{\nu_f}\int\delta U_f^*\left\{U_f-H_{fi}+i\pi\sum_{\nu_{f'}}\int H_{ff'}U_{f'}\rho_{f'}d\Omega_{f'}\right\}\rho_f d\Omega_f=0.$$

The Ritz method is applied to its solution; that is, trial functions are introduced for U_f^* and the result minimized. The authors give two examples, both of which show fair agreement with exact calculations. *H. Feshbach.*

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